



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>





600003880P

✓
A

S.H. 1828.

TREATISE

ON THE

GEOMETRICAL REPRESENTATION

OF THE

SQUARE ROOTS OF NEGATIVE QUANTITIES.



BY THE

REV. JOHN WARREN, A.M.

FELLOW AND TUTOR OF JESUS COLLEGE, CAMBRIDGE.

CAMBRIDGE:

Printed by J. Smith, Printer to the University;

SOLD BY T. STEVENSON, AND J. & J. DEIGHTON, CAMBRIDGE;
BALDWIN & CRADOCK, LONDON; PARKER, OXFORD;
BELL & BRADFUTE, EDINBURGH; MILLIKEN
AND W. F. WAKEMAN, DUBLIN.

1828.

936.

The Author begs leave to acknowledge his obligations to the SYNDICS of the University Press; who have, from the funds at their disposal, contributed liberally to the expense of this Work.

CONTENTS.

CHAP. I.

	Page
Definitions. Addition, Subtraction, Proportion, Multiplication, Division. Fractions, and raising of Powers	1

CHAP. II.

Roots of Quantities, Fractional and Negative Indices	24
--	----

CHAP. III.

Binomial Theorem. Expansion of a^x in a series arranged according to the powers of x . Differentiation of a^x ...	76
---	----

CHAP. IV.

Examples illustrative of the Principles established in the preceding Chapters	102
---	-----

A TREATISE
ON THE
GEOMETRICAL REPRESENTATION
OF THE
SQUARE ROOTS OF NEGATIVE QUANTITIES,

CHAP. I.

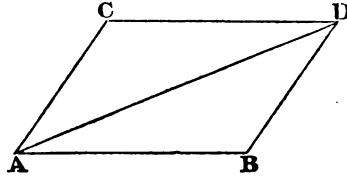
**DEFINITIONS, ADDITION, SUBTRACTION, PROPORTION,
MULTIPLICATION, DIVISION, FRACTIONS, AND RAISING
OF POWERS.**

(ART. 1.) **ALL** straight lines drawn in a given plane, from a given point, are represented in *length* and *direction* by Algebraic quantities; and in the following Treatise whenever the word *quantity* is used, it is to be understood as signifying a *line*.

(2.) **DEF.** The given point from which the straight lines are measured is called the *origin*.

(3.) **DEF.** The *sum* of two quantities is the diagonal of the parallelogram whose sides are the two quantities.

Thus if a represent AB in length and direction,

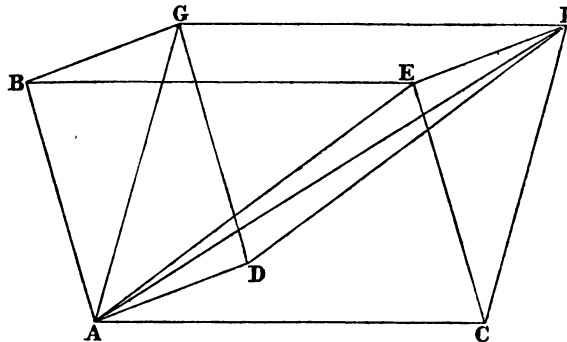


and b represent AC in length and direction; and the parallelogram $ABDC$ be completed, and the diagonal AD be drawn; $a + b$ represents AD in length and direction.

(4.) DEF. *Subtraction* is the reverse of Addition, or if the sum of any two quantities a, b be c ; b is called the *difference* which arises from *subtracting* a from c .

(5.) *If three quantities be added together, the sum will be the same, whatever be the order in which the quantities are added together.*

Let a, b, c be the three quantities.



Let $AB = a, AC = b, AD = c$.

Let the parallelogram $ABEC$ be completed, and the diagonal AE be drawn

$$AE = a + b.$$

Let the parallelogram $Aefd$ be completed, and the diagonal AF be drawn

$$AF = (a + b) + c.$$

Again, let the parallelogram $ABGD$ be completed, and the diagonal AG be drawn

$$AG = a + c.$$

Join CF , GF ;

$\therefore BG$ is parallel and equal in length to AD ,

and EF also parallel and equal in length to AD ,

BG is parallel and equal in length to EF ;

$\therefore GF$ is parallel and equal in length to BE , and therefore to AC ;

$\therefore ACFG$ is a parallelogram, and AF the diagonal;

$$\therefore AF = (a + c) + b;$$

$$\therefore (a + b) + c = (a + c) + b.$$

In like manner $(a + b) + c = (b + c) + a$.

(6.) *The sum of any number of quantities is the same, whatever be the order in which the quantities are added together.*

First, let there be four quantities, a, b, c, d .

By the preceding article, the sum of any three of these will be the same, whatever be the order in which they are added together.

$$\text{Let } a + b + c = s,$$

$$a + b + d = t,$$

$$a + c + d = u,$$

$$b + c + d = v.$$

To prove that $s + d = t + c = u + b = v + a$,

$$s = a + b + c$$

$$= (a + b) + c;$$

$$\therefore s + d = \{(a + b) + c\} + d$$

$$= \{(a + b) + d\} + c, \quad (\text{by Art. 5.})$$

$$= t + c.$$

In the same manner it may be proved that

$$s + d = u + b = v + a;$$

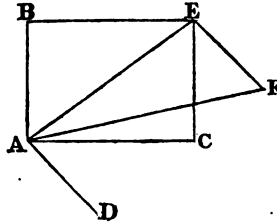
therefore if four quantities be added together, the sum will be the same, whatever be the order in which the quantities are added together.

In like manner it may be proved, if there be five quantities, &c.; therefore the sum of any number of quantities is the same, whatever be the order in which the quantities are added together.

(7.) *Let there be any number of quantities AB, AC, AD; and let BE be parallel and equal in length to AC, and EF parallel and equal in length to AD, and let AF be joined; AF = AB + AC + AD.*

For join *AE, CE*.

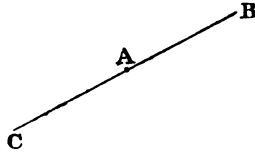
Then $ABEC$ is a parallelogram, and AE the diagonal;



$$\therefore AE = AB + AC.$$

In like manner it may be proved that $AF = AE + AD$;

$$\therefore AF = AB + AC + AD.$$



(8.) **DEF.** If a represent a quantity in any direction, $-a$ will represent a quantity equal in length to the former, but drawn in the *opposite* direction: thus if AB be represented by a , and BA be produced to C , and AC taken equal in length to AB , AC will be represented by $-a$.

(9.) *The difference which arises from subtracting b from a is $a + (-b)$.*

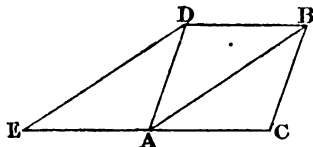
Let $AB = a$, $AC = b$.

Join BC , and complete the parallelogram $ACBD$,

$$AB = AC + AD,$$

$$\text{or } a = b + AD;$$

$\therefore AD$ is the difference which arises from subtracting b from a .



Again, produce CA to E , and make AE equal in length to AC , and join ED ; therefore since DB is parallel and equal in length to AC , it is also parallel and equal in length to EA ; therefore $EDBA$ is a parallelogram, and AD the diagonal;

$$\begin{aligned}\therefore AD &= AB + AE \\ &= a + (-b); \end{aligned}$$

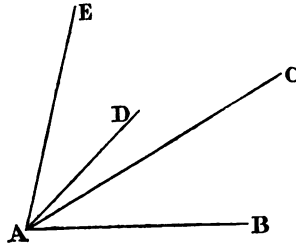
therefore the difference which arises from subtracting b from $a = a + (-b)$.

(10.) DEF. $a + (-b)$ is expressed $a - b$.

(11.) DEF. Quantities drawn from the origin in a certain direction, which direction is arbitrarily assumed, are called *positive* quantities; and those drawn in the *opposite* direction are called *negative* quantities.

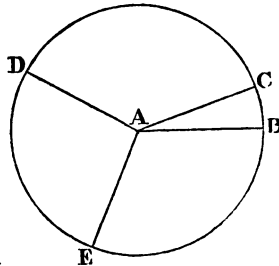
(12.) DEF. The first of four quantities is said to have to the second the same ratio which the third has to the fourth; when the first has *in length* to the second the same ratio which the third has *in length* to the fourth, according to Euclid's definition; and also the angle at which the fourth is inclined to the

third, is equal to the angle at which the second is inclined to the first, and is measured in the same direction*.



Thus let AB in length : $AC :: AD$ in length : AE , and also let angle DAE be equal to angle BAC , and be measured in the same direction from AD ,

* Angles are said to be measured in the same direction, when the arcs, which subtend them, are measured in the same direction round the circle; and are said to be measured in opposite directions, when the arcs are measured in opposite directions.



Thus angle DAE is said to be measured from AD in the same direction that angle BAC is from AB , because arc DE is measured from D in the same direction round the circle that arc BC is from B ; and angle EAD is said to be measured from AE in the opposite direction that angle BAC is from AB , because arc ED is measured from E in the opposite direction that arc BC is from B .

that angle BAC is from AB : then AB is said to have to AC the same ratio which AD has to AE .

(13.) *If d be a fourth proportional to a, b, c ; there is no other quantity different from d which is also a fourth proportional to a, b, c .*

For, (by Euclid, Book v. Props. 8 and 13.)

a has in length a greater ratio to b than c has in length to any quantity greater than d ; and c has in length a greater ratio to any quantity less than d , than a has in length to b ; therefore there is no quantity greater or less in length than d , which is a fourth proportional to a, b, c .

Neither is there any quantity different in direction from d , which is a fourth proportional to a, b, c .

For since d is a fourth proportional to a, b, c ; d is inclined at the same angle to c , which b is to a .

Let this angle be A .

Since b is inclined to a at the angle A , it is also inclined to a at the angles $A + 360^\circ$, $A + 2.360^\circ$, &c.

$A - 360^\circ$, $A - 2.360^\circ$, &c.;

therefore any quantity which is equal in length to d , and inclined to c at any of the angles,

A , $A + 360^\circ$, $A + 2.360^\circ$, &c.

$A - 360^\circ$, $A - 2.360^\circ$, &c.

is by the definition a fourth proportional to a, b, c .

But since d is inclined to c at the angle A , it is also inclined to c at the angles

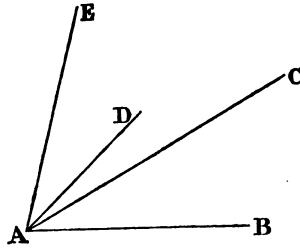
$$A + 360^\circ, A + 2.360^\circ, \&c.$$

$$A - 360^\circ, A - 2.360^\circ, \&c.;$$

therefore there is no quantity different *in direction* from d , which is a fourth proportional to a, b, c ;

therefore there is no quantity different from d , which is a fourth proportional to a, b, c .

(14.) *If four quantities be proportionals, they are proportionals also when taken inversely or alternately.*



Let $AB : AC :: AD : AE$.

Then inversely $AC : AB :: AE : AD$,

and alternately $AB : AD :: AC : AE$.

For since

AB in length : $AC :: AD$ in length : AE ,

(by Simson's Euclid, Book v. Prop. B.)

AC in length : $AB :: AE$ in length : AD .

And since angle DAE is equal to angle BAC , and is measured in the same direction from AD , that BAC is from AB ; angle EAD is equal to angle CAB , and

is measured in the same direction from AE , that CAB is from AC ;

$$\therefore AC : AB :: AE : AD.$$

Also since

$$AB \text{ in length} : AC :: AD \text{ in length} : AE,$$

(by Euclid, Book v. Prop. 16.)

$$AB \text{ in length} : AD :: AC \text{ in length} : AE.$$

And since angle $DAE = \text{angle } BAC$, angle $CAE = \text{angle } BAD$;

$$\therefore AB : AD :: AC : AE.$$

(15.) *If* $a : b :: c : d$, *and* $c : d :: e : f$; *then*
 $a : b :: e : f$.

For since $a \text{ in length} : b :: c \text{ in length} : d$,
and $c \text{ in length} : d :: e \text{ in length} : f$,

(by Euclid, Book v. Prop. 11.)

$$a \text{ in length} : b :: e \text{ in length} : f.$$

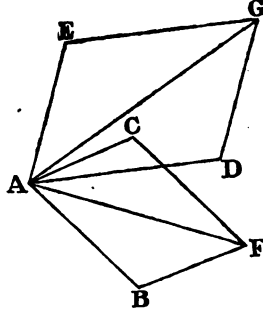
And since the angle at which f is inclined to e is equal to the angle at which d is inclined to c , and the angle at which d is inclined to c equal to the angle at which b is inclined to a ; the angle at which f is inclined to e is equal to the angle at which b is inclined to a ;

$$\therefore a : b :: e : f.$$

(16.) *If* $a : b :: c : d$, *then* $a + b : b :: c + d : d$.

Let $AB = a$, $AC = b$, $AD = c$, $AE = d$.

Complete the parallelograms $ABFC$, $ADGE$, and draw the diagonals AF , AG .



Then $AF = a + b$, and $AG = c + d$.

And since AB in length : $AC :: AD$ in length : AE ,

and angle $DAE = \text{angle } BAC$,

parallelogram $BACF$ is similar to parallelogram $DAEG$;

therefore triangle FAC is similar to triangle GAE ,

therefore angle $FAC = GAE$,

and AF in length : $AC :: AG$ in length : AE ;

$\therefore AF : AC :: AG : AE$,

or $a + b : b :: c + d : d$.

(17.) DEF. Unity is a positive quantity arbitrarily assumed, from a comparison with which the values of other quantities are determined.

(18.) DEF. If there be three quantities such that, unity is to the first as the second to the third; the third is called the *product* which arises from the *multiplication* of the second by the first.

Thus if, $1 : a :: b : c$, c is called the product which arises from the multiplication of b by a .

(19.) *If c = the product which arises from the multiplication of b by a , c is also equal to the product which arises from the multiplication of a by b .*

For since $1 : a :: b : c$,

alternando $1 : b :: a : c$;

therefore c is also the product which arises from the multiplication of a by b .

(20.) $a : ac :: b : bc$.

For (by definition of multiplication)

$1 : c :: a : ac$,

and $1 : c :: b : bc$;

\therefore (by Art. 15.) $a : ac :: b : bc$.

(21.) COR. Hence alternando

$a : b :: ac : bc$.

(22.) *If three quantities be multiplied together, the product will be the same, whatever be the order in which they are multiplied together.*

Let a, b, c be the three quantities.

Let $ab = d$, and $cd = e$,

and let $ac = f$, and $bf = g$.

To prove that $e = g$,

$1 : a :: b : d$,

and $1 : a :: c : f$;

$$\begin{aligned} \therefore b : d &:: c : f \\ &:: bc : bf \text{ (Art. 21.)} \\ &:: bc : g; \end{aligned}$$

$$\begin{aligned} \text{also } b : d &:: bc : dc \text{ (Art. 21.)} \\ &:: bc : e; \end{aligned}$$

$$\therefore \text{ (by Art. 13.) } g = e,$$

$$\text{or } (a \cdot c) \cdot b = (a \cdot b) \cdot c.$$

In like manner it may be proved that

$$(a \cdot c) \cdot b = (b \cdot c) \cdot a;$$

therefore the product is the same, whatever be the order, in which the quantities are multiplied together.

(23.) *Cor. Hence the product of any number of quantities is the same, whatever be the order, in which the quantities are multiplied together.*

$$(24.) \quad (a + b) \cdot c = ac + bc.$$

$$\text{For (by Art. 21.) } a : b :: ac : bc;$$

$$\therefore \text{ (by Art. 16.) } a + b : b :: ac + bc : bc;$$

$$\therefore \text{ alternando } a + b : ac + bc :: b : bc.$$

But by definition of multiplication

$$1 : c :: b : bc;$$

$$\therefore 1 : c :: a + b : ac + bc;$$

$$\therefore (a + b) c = ac + bc.$$

$$(25.) \quad (a + b + c) d = ad + bd + cd.$$

$$\begin{aligned} \text{For (by preceding Article) } (a + b + c) d &= (a + b) d + cd \\ &= ad + bd + cd. \end{aligned}$$

$$(26.) \quad (a + b) \cdot (c + d) = ac + ad + bc + bd.$$

$$\begin{aligned} \text{For } (a + b) \cdot (c + d) &= a \cdot (c + d) + b \cdot (c + d) \\ &= ac + ad + bc + bd. \end{aligned}$$

$$(27.) \quad a \times (-b) = -ab.$$

$$\begin{aligned} \text{For let } ab &= x, \\ 1 : a &:: b : x. \end{aligned}$$

But since the angle at which $-x$ is inclined to $-b$ is equal to the angle at which x is inclined to b ,

$$\begin{aligned} b : x &:: -b : -x; \\ \therefore 1 : a &:: -b : -x; \\ \therefore a \times (-b) &= -x = -ab. \end{aligned}$$

$$(28.) \quad (-a) \cdot (-b) = ab.$$

For by the preceding article

$$\begin{aligned} (-a) \cdot (-b) &= -(-a)b \\ &= -(-ab) \\ &= ab. \end{aligned}$$

(29.) **DEF.** If three quantities be such that the first is to unity as the second is to the third; the first quantity is called the *quotient* which arises from the *division* of the second by the third.

Thus if $c : 1 :: a : b$; c is called the quotient which arises from the division of a by b .

(30.) DEF. The quotient which arises from the division of a by b is thus expressed $a \div b$,

or thus $\frac{a}{b}$.

(31.) If $c = \frac{a}{b}$, then $a = bc$ and conversely.

Let $c = \frac{a}{b}$;

$\therefore c : 1 :: a : b$;

\therefore invertendo $1 : c :: b : a$;

$\therefore a = bc$.

In like manner the converse may be proved.

(32.) $\frac{-a}{b} = -\left(\frac{a}{b}\right)$.

For let $\frac{a}{b} = c$;

then $c : 1 :: a : b$;

\therefore alternando $c : a :: 1 : b$;

$\therefore -c : -a :: 1 : b$;

\therefore alternando $-c : 1 :: -a : b$;

$\therefore \frac{-a}{b} = -c = -\left(\frac{a}{b}\right)$.

(33.) $\frac{a}{-b} = -\left(\frac{a}{b}\right)$.

Let $\frac{a}{b} = c$,

$$\begin{aligned}
c : 1 &:: a : b \\
&:: -a : -b \\
\therefore c : -a &:: 1 : -b; \\
\therefore -c : a &:: 1 : -b; \\
\therefore -c : 1 &:: a : -b; \\
\therefore \frac{a}{-b} &= -c = -\left(\frac{a}{b}\right).
\end{aligned}$$

$$(34.) \quad \frac{-a}{-b} = \frac{a}{b}.$$

$$\text{Let } \frac{a}{b} = c,$$

$$\begin{aligned}
c : 1 &:: a : b \\
&:: -a : -b; \\
\therefore \frac{-a}{-b} &= c = \frac{a}{b}.
\end{aligned}$$

$$(35.) \quad \text{If } a : b :: c : d, \text{ then } \frac{a}{b} = \frac{c}{d}.$$

$$\text{Let } \frac{a}{b} = x,$$

$$\begin{aligned}
x : 1 &:: a : b \\
&:: c : d;
\end{aligned}$$

$$\therefore \frac{c}{d} = x = \frac{a}{b}.$$

$$(36.) \quad \text{If } \frac{a}{b} = \frac{c}{d}, \quad a : b :: c : d.$$

$$\text{For let } \frac{a}{b} \text{ or } \frac{c}{d} = x,$$

$$x : 1 :: a : b,$$

$$\text{also } x : 1 :: c : d;$$

$$\therefore a : b :: c : d.$$

$$(37.) \quad \frac{a}{b} = \frac{ac}{bc}.$$

$$\text{For (by Art. 21.) } a : b :: ac : bc;$$

$$\therefore (\text{by Art. 35.}) \quad \frac{a}{b} = \frac{ac}{bc}.$$

$$(38.) \quad \frac{b}{a} + \frac{c}{a} = \frac{b+c}{a}.$$

$$\text{For let } \frac{b}{a} = x, \quad \frac{c}{a} = y;$$

$$\therefore (\text{by Art. 31.}) \quad b = ax, \text{ and } c = ay;$$

$$\therefore b + c = ax + ay = (\text{by Art. 24.}) \quad a(x + y);$$

$$\therefore \frac{b+c}{a} = x + y = \frac{b}{a} + \frac{c}{a}.$$

$$(39.) \quad \text{COR. Hence } \frac{b}{a} + \frac{d}{c} = \frac{bc + ad}{ac}.$$

$$\text{For } \frac{b}{a} = \frac{bc}{ac}, \text{ and } \frac{d}{c} = \frac{ad}{ac};$$

$$\therefore \frac{b}{a} + \frac{d}{c} = \frac{bc + ad}{ac}.$$

$$(40.) \quad \frac{b}{a} \times \frac{d}{c} = \frac{bd}{ac}.$$

For let $\frac{b}{a} = x$, $\frac{d}{c} = y$;

then $b = ax$ and $d = cy$;

$\therefore bd = ax \times cy = ac \times xy$, (by Art. 23.)

$$\therefore \frac{bd}{ac} = xy = \frac{b}{a} \times \frac{d}{c}.$$

$$(41.) \quad \frac{b}{a} \div \frac{d}{c} = \frac{bc}{ad}.$$

For let $\frac{b}{a} = x$, $\frac{d}{c} = y$;

then $b = ax$, and $d = cy$;

$\therefore bc = (ax)c = (ac)x$,

and $ad = a(cy) = (ac)y$;

$$\therefore \frac{bc}{ad} = \frac{(ac)x}{(ac)y} = \frac{x}{y} = \frac{b}{a} \div \frac{d}{c}.$$

(42.) **DEF.** Any quantity $a \times a \times a$, &c. to n factors is called the n^{th} power of a , and expressed a^n .

(43.) *If b be an n^{th} power of a , b is the only n^{th} power of a .*

For $1 : a :: a : a^2$;

therefore since a^2 is a fourth proportional to 1, a , a ;
it is the only fourth proportional to 1, a , a ;

therefore there is only one second power of a .

In like manner since a^3 is a fourth proportional to 1, a , a^2 ; it is the only fourth proportional to 1, a , a^2 ;

therefore there is only one-third power of a ;

therefore only one-fourth power of a , &c.;

therefore only one n^{th} power of a .

$$(44.) \quad a^m \times a^n = a^{m+n}.$$

For $a^m = a \times a \times a \dots$ to m factors,

and $a^n = a \times a \times a \dots$ to n factors;

$$\therefore a^m \times a^n = a \times a \times a \dots \text{to } m+n \text{ factors} \\ = a^{m+n}.$$

$$(45.) \quad \frac{a^m}{a^n} = a^{m-n}, \text{ when } m \text{ is greater than } n;$$

$$\text{and } \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \text{ when } m \text{ is less than } n.$$

For, when m is greater than n

$$(\text{by Art. 44}) \quad a^n \times a^{m-n} = a^{n+m-n} = a^m;$$

$$\therefore (\text{by Art. 31.}) \quad \frac{a^m}{a^n} = a^{m-n}.$$

Again when m is less than n , $a^m \times a^{n-m} = a^n$;

$$\therefore \frac{a^m}{a^n} = \frac{a^m}{a^m \times a^{n-m}} = (\text{by Art. 37.}) \frac{1}{a^{n-m}}.$$

$$(46.) \quad (ab)^m = a^m \cdot b^m.$$

For $(ab)^m = (ab) \cdot (ab) \cdot (ab) \dots$ to m factors
 $= a \cdot a \cdot a \dots$ to m factors $\times b \cdot b \cdot b \dots$ to m factors
 $= a^m \cdot b^m.$

$$(47.) \quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

$$\begin{aligned} \text{For } \left(\frac{a}{b}\right)^m &= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \dots \text{to } m \text{ factors} \\ &= \frac{a \times a \times a \dots \text{to } m \text{ factors}}{b \times b \times b \dots \text{to } m \text{ factors}} \\ &= \frac{a^m}{b^m}. \end{aligned}$$

$$(48.) \quad (a^m)^n = a^{mn}.$$

For $(a^m)^n = a^m \times a^m \times a^m \dots$ to n factors
 $= a^{mn}, (\text{by Art. 44.})$

(49.) *If a and b be positive quantities and a greater than b , a^m is greater than b^m .*

Let $a = b + c$,

$$a^2 = b^2 + 2bc + c^2;$$

$\therefore a^2$ is greater than b^2 .

Let $a^2 = b^2 + d$,

$$a^3 = a \cdot (b^2 + d) = (b + c) \cdot (b^2 + d) = b^3 + b^2c + bd + cd;$$

$\therefore a^3$ is greater than b^3 .

In like manner it may be proved that a^4 is greater than b^4 , &c.;

$\therefore a^m$ is greater than b^m .

(50.) *If a and b be any quantities, and a in length greater than b, a^m is in length greater than b^m .*

Let c be a positive quantity equal in length to a ,
and $d \dots \dots \dots b$;

c is greater than d ;

therefore by the preceding Article c^m is greater than d^m ;

but c^m is in length equal to a^m , and d^m to b^m ;

$\therefore a^m$ is in length greater than b^m .

(51.) *If $ab = c$, and a be inclined to unity at an angle = A, and b at an angle = B; c will be inclined to unity at an angle = A + B.*

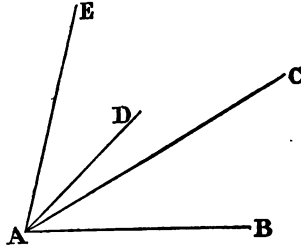
For let $AB = \text{unity}$, $AC = a$, $AD = b$, $AE = c$,

then angle $BAC = A$, angle $BAD = B$;

and since $c = ab$,

$$1 : a :: b : c;$$

\therefore (by Art. 12.) angle $DAE = \text{angle } BAC = A$;
and angle $BAD = B$;



therefore angle $BAE = A + B$;

therefore c is inclined to unity at an angle $= A + B$.

(52.) If $b = a^m$, and a be inclined to unity at an angle $= A$; b will be inclined to unity at an angle $= mA$.

For since $a^2 = a \cdot a$,

(by Art. 51.) a^2 is inclined to unity at an angle
 $= A + A = 2A$;

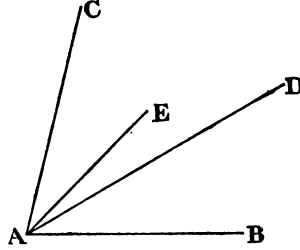
and since $a^3 = a \cdot a^2$, a^3 is inclined to unity at an angle
 $= A + 2A = 3A$;

in like manner a^4 is inclined to unity at an angle
 $= 4A$; &c.;

$\therefore b = a^m$ is inclined to unity at an angle $= mA$.

(53.) If a be inclined to unity at an angle
 $= A$, and b at an angle $= B$, and $c = \frac{a}{b}$; c will
be inclined to unity at an angle $= A - B$.

Let $AB = \text{unity}$, angle $BAC = A$, angle $BAD = B$,
 $AC = a$, $AD = b$, $AE = c$;



$$\text{since } c = \frac{a}{b},$$

$$c : 1 : a : b;$$

therefore angle $CAD = \text{angle } EAB$;

therefore angle $BAE = \text{angle } BAC - \text{angle } BAD$
 $= A - B$;

therefore c is inclined to unity at an angle $= A - B$.



CHAP. II.

ROOTS OF QUANTITIES, FRACTIONAL AND NEGATIVE INDICES.

(ART. 54.) DEF. ANY quantity, which when raised to the n^{th} power produces a quantity a , is called the n^{th} root of a , and expressed thus $\sqrt[n]{a}$.

(55.) DEF. The m^{th} power of the n^{th} root of a , or $(\sqrt[n]{a})^m$ is expressed thus $a^{\frac{m}{n}}$.*

(56.) DEF. $\frac{1}{a^m}$ is expressed thus a^{-m} , where m may be either whole or fractional.

* This as a general definition of a power of any quantity is defective, for, if the p^{th} power of a be required, where p is a surd, since no fraction, whose numerator and denominator are whole numbers, will accurately represent p , this definition cannot strictly be applied. But, as the fifth definition of the fifth Book of Euclid is a correct and general definition of proportion of magnitudes, and that usually given by Algebraists a defective one, so a general definition of a power of any quantity might have been given, bearing the same analogy to the fifth definition of the fifth Book of Euclid, which the definition in Article 55 bears to the common Algebraic definition of proportion. Nevertheless, as the demonstration of the propositions, necessary to establish the properties of powers of quantities, would by this definition have been rendered tedious: the definition in Art. 55, being one in use amongst Algebraists, has been adopted, notwithstanding its imperfection as a general definition.

(57.) *If b be the n^{th} root of a , and d the n^{th} root of c , and c be in length equal to a ; then shall d be in length equal to b .*

For if not, let one of them, d , be the greater in length.

Then since d is greater in length than b ; (by Art. 50.) d^n is in length greater than b^n ;

But $d^n = c$, and $b^n = a$;

therefore c is in length greater than a , which is contrary to the hypothesis;

therefore d is in length equal to b .

(58.) *COR. Hence all the n^{th} roots of any quantity a are in length equal to one another.*

(59.) *If a be in length equal to b , $a^{\frac{m}{n}}$ is in length equal to $b^{\frac{m}{n}}$.*

For, by the preceding Article,

$\sqrt[n]{a}$ is in length equal to $\sqrt[n]{b}$;

$\therefore (\sqrt[n]{a})^m$ is in length equal to $(\sqrt[n]{b})^m$,

that is, $a^{\frac{m}{n}}$ is in length equal to $b^{\frac{m}{n}}$.

(60.) *If a be in length equal to b , $a^{-\frac{m}{n}}$ is in length equal to $b^{-\frac{m}{n}}$.*

For $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$, and $b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$;

$$\therefore a^{-\frac{n}{2}} : b^{-\frac{n}{2}} :: \frac{1}{a^{\frac{n}{2}}} : \frac{1}{b^{\frac{n}{2}}} :: b^{\frac{n}{2}} : a^{\frac{n}{2}};$$

and by the preceding Article $b^{\frac{n}{2}}$ is in length equal to $a^{\frac{n}{2}}$;

$$\therefore a^{-\frac{n}{2}} \text{ is in length equal to } b^{-\frac{n}{2}}.$$

(61.) If $b = \sqrt[n]{a}$, b has n different values.

For let a be inclined to unity at an angle $= A$.

Then a is also inclined to unity at angles

$$= A + p \cdot 360^\circ,$$

where p is any whole number, either positive or negative.

Now in order that b may be an n^{th} root of a , it is necessary that b should be inclined to unity at an angle B , such that

$$nB = A + p \cdot 360^\circ \text{ (by Art. 52.)};$$

$$\therefore B = \frac{A + p \cdot 360^\circ}{n}.$$

For p substitute successively 0, 1, 2, &c.

$$n - 1, n, n + 1, \text{ \&c.}$$

$$\text{also } -1, -2, \text{ \&c.}$$

The corresponding values of B will be respectively

$$\frac{A}{n}, \frac{A + 360^\circ}{n}, \frac{A + 2 \cdot 360^\circ}{n}, \text{ \&c.,}$$

$$\frac{A + \overline{n-1} \cdot 360^\circ}{n}, \frac{A + n \cdot 360^\circ}{n}, \frac{A + \overline{n+1} \cdot 360^\circ}{n}, \&c.,$$

$$\frac{A - 360^\circ}{n}, \frac{A - 2 \cdot 360^\circ}{n}, \&c.$$

$$\text{But } \frac{A + n \cdot 360^\circ}{n} = \frac{A}{n} + 360^\circ,$$

$$\text{and } \frac{A + \overline{n+1} \cdot 360^\circ}{n} = \frac{A + 360^\circ}{n} + 360^\circ;$$

therefore the value of b , which is inclined to unity at an angle $= \frac{A}{n}$, is also inclined to unity at an angle

$$= \frac{A}{n} + 360^\circ, \text{ or } \frac{A + n \cdot 360^\circ}{n},$$

and all the values of b are in length equal to one another (by Art. 58.); therefore the value of b , which is inclined to unity at an angle $= \frac{A}{n}$, coincides with the value of b which is inclined to unity at an angle $= \frac{A + n \cdot 360^\circ}{n}$, and therefore is equal to it.

In like manner the value of b which is inclined to unity at an angle $= \frac{A + 360^\circ}{n}$, is equal to the value of b which is inclined to unity at an angle

$$= \frac{A + \overline{n+1} \cdot 360^\circ}{n}.$$

Also since $\frac{A - 360^\circ}{n} = \frac{A + \overline{n - 1} \cdot 360^\circ}{n} - 360^\circ,$

and $\frac{A - 2 \cdot 360^\circ}{n} = \frac{A + \overline{n - 2} \cdot 360^\circ}{n} - 360^\circ.$

The value of b , which is inclined to unity at an angle $= \frac{A + \overline{n - 1} \cdot 360^\circ}{n}$, is equal to the value of b , which is inclined to unity at an angle $= \frac{A - 360^\circ}{n}$; and the value of b , which is inclined to unity at an angle $= \frac{A + \overline{n - 2} \cdot 360^\circ}{n}$, is equal to the value of b , which is inclined to unity at an angle $= \frac{A - 2 \cdot 360^\circ}{n}$; &c.;

therefore b has n different values.

(62.) DEF. If a be inclined to unity at an angle $= A$, A being positive and less than 360° ; a_0 expresses a considered as inclined to unity at an angle $= A$,

$a_1 \dots \dots \dots$ at an angle $= A + 360^\circ,$

$a_2 \dots \dots \dots = A + 2 \cdot 360^\circ,$

&c. &c.

also $a_{-1} \dots \dots \dots = A - 360^\circ,$

$a_{-2} \dots \dots \dots$ at an angle $= A - 2.360^\circ$,

&c.

&c.

and generally $a_p \dots \dots \dots = A + p.360^\circ$,

where $p = 0$, or any whole number, either positive or negative.

Thus $\sqrt[p]{a}$ expresses that value of $\sqrt[p]{a}$, which arises from considering a as inclined to unity at the angle $= A$, $\sqrt[1]{a}$ expresses that value of $\sqrt[p]{a}$ which arises from considering a as inclined to unity at the angle $= A + 360^\circ$, and generally $\sqrt[p]{a}$ expresses that value of $\sqrt[p]{a}$ which arises from considering a as inclined to unity at the angle $= A + p.360^\circ$.

(63.) If $(a_p)^{\frac{m}{n}} = b$, b is inclined to unity at an angle $= \frac{m}{n} (A + p.360^\circ)$.

For let $\sqrt[p]{a} = c$, then $c^m = b$ (by Art. 55.)

(By Art. 61, 62.) c is inclined to unity at an angle

$$= \frac{A + p.360^\circ}{n};$$

$$\text{let } \frac{A + p.360^\circ}{n} = C;$$

then since $b = c^m$, b is inclined to unity at an angle

$$= mC$$

$$= \frac{m}{n} \cdot (A + p \cdot 360^\circ).$$

(64.) If $(a_p)^{-\frac{m}{n}} = b$, b is inclined to unity at an angle $= -\frac{m}{n} \cdot (A + p \cdot 360^\circ)$.

$$b = \frac{1}{(a_p)^{\frac{m}{n}}},$$

and 1 is inclined to unity at an angle $= 0$,

and $(a_p)^{\frac{m}{n}}$ is inclined to unity at an angle

$$= \frac{m}{n} (A + p \cdot 360^\circ) \text{ (by Art. 63.)}$$

therefore b is inclined to unity at an angle

$$= 0 - \frac{m}{n} (A + p \cdot 360^\circ) \text{ (by Art. 53.)}$$

$$= -\frac{m}{n} \cdot (A + p \cdot 360^\circ).$$

(65.) COR. In the two preceding Articles if a be a positive quantity, $A = 0$; therefore b is inclined to unity at an angle $= \pm \frac{m}{n} p \cdot 360^\circ$.

In this case; if $p = 0$, b is inclined to unity at an angle $= 0$, that is, is a positive quantity;

If $p = 1$, b is inclined to unity at an angle

$$= \pm \frac{m}{n} 360^\circ.$$

(66.) $\left(a\right)_0^{\frac{km}{kn}} = \left(a\right)_0^{\frac{m}{n}}$, a being a positive quantity.

For let $\sqrt[n]{a} = b$, and $\sqrt[k]{b} = c$,

then (by Art. 65.) b and c are positive quantities ;

$$\text{And } a = b^n, \quad b = c^k,$$

$$\therefore a = (c^k)^n = (\text{by Art. 48.}) c^{kn},$$

$$\therefore c \text{ is a value of } \sqrt[kn]{a};$$

And c is a positive quantity,

$$\therefore c = \sqrt[kn]{a},$$

$$\therefore c^k = \left(\sqrt[kn]{a}\right)^k,$$

$$\therefore b = \left(\sqrt[kn]{a}\right)^k,$$

$$\therefore b^m = \left(\sqrt[kn]{a}\right)^{km} = \left(a\right)_0^{\frac{km}{kn}};$$

$$\text{But } b^m = \left(a\right)_0^{\frac{m}{n}} \text{ (by Art. 55.)}$$

$$\therefore \left(a\right)_0^{\frac{km}{kn}} = \left(a\right)_0^{\frac{m}{n}}.$$

(67.) $\binom{a}{0}^{\frac{n}{q}} \times \binom{a}{0}^{\frac{p}{q}} = \binom{a}{0}^{\frac{mq+np}{nq}}$, *a being a positive quantity.*

$$\text{For (by Art. 66.) } \binom{a}{0}^{\frac{n}{q}} = \binom{a}{0}^{\frac{nq}{nq}} = \left(\sqrt[nq]{a}\right)^{nq}$$

$$\binom{a}{0}^{\frac{p}{q}} = \binom{a}{0}^{\frac{np}{nq}} = \left(\sqrt[nq]{a}\right)^{np};$$

$$\begin{aligned} \therefore \text{(by Art. 44.) } \binom{a}{0}^{\frac{n}{q}} \times \binom{a}{0}^{\frac{p}{q}} &= \left(\sqrt[nq]{a}\right)^{mq+np} \\ &= \binom{a}{0}^{\frac{mq+np}{nq}}. \end{aligned}$$

(68.) $\binom{a}{0}^{\frac{n}{q}} \times \binom{a}{0}^{\frac{-p}{q}} = \binom{a}{0}^{\frac{mq-np}{nq}}$, *a being a positive quantity.*

First let mq be greater than np ,

$$\begin{aligned} \binom{a}{0}^{\frac{n}{q}} \times \binom{a}{0}^{\frac{-p}{q}} &= \frac{\binom{a}{0}^{\frac{n}{q}}}{\binom{a}{0}^{\frac{p}{q}}} = \frac{\binom{a}{0}^{\frac{nq}{nq}}}{\binom{a}{0}^{\frac{np}{nq}}} = \frac{\left(\sqrt[nq]{a}\right)^{nq}}{\left(\sqrt[nq]{a}\right)^{np}} \\ &= \left(\sqrt[nq]{a}\right)^{mq-np} \text{ (by Art. 45.)} \\ &= \binom{a}{0}^{\frac{mq-np}{nq}}. \end{aligned}$$

Next let mq be less than np ,

$$\binom{a}{0}^{\frac{n}{q}} \times \binom{a}{0}^{\frac{-p}{q}} = \frac{1}{\left(\sqrt[nq]{a}\right)^{np-mq}} \text{ (by Art. 45.)}$$

$$\begin{aligned} \binom{a}{0}^{\frac{m}{q}} \times \binom{a}{0}^{-\frac{p}{q}} &= \frac{1}{\binom{a}{0}^{\frac{np-mq}{q}}} = \binom{a}{0}^{-\frac{np-mq}{q}} \\ &= \binom{a}{0}^{\frac{mq-np}{q}}. \end{aligned}$$

(69.) $\binom{a}{0}^{-\frac{m}{q}} \times \binom{a}{0}^{-\frac{p}{q}} = \binom{a}{0}^{-\frac{mq-np}{q}}$, *a being a positive quantity.*

$$\begin{aligned} \text{For } \binom{a}{0}^{-\frac{m}{q}} \times \binom{a}{0}^{-\frac{p}{q}} &= \frac{1}{\binom{a}{0}^{\frac{m}{q}}} \times \frac{1}{\binom{a}{0}^{\frac{p}{q}}} = \frac{1}{\binom{a}{0}^{\frac{m}{q}} \times \binom{a}{0}^{\frac{p}{q}}} \\ &= \frac{1}{\binom{a}{0}^{\frac{mq+np}{q}}} \text{ (by Art. 67.)} \\ &= \binom{a}{0}^{-\frac{mq+np}{q}}. \end{aligned}$$

(70.) $\frac{\binom{a}{0}^m}{\binom{a}{0}^n} = \binom{a}{0}^{m-n}$, *where a is a positive quantity, and m and n either whole or fractional, positive or negative.*

$$\begin{aligned} \text{For } \frac{\binom{a}{0}^m}{\binom{a}{0}^n} &= \binom{a}{0}^m \times \binom{a}{0}^{-n} \\ &= \binom{a}{0}^{m-n} \text{ (by Art. 67, 68, 69).} \end{aligned}$$

(71.) $\left(\frac{a}{0}\right)^{\frac{1}{n}} \times \left(\frac{b}{0}\right)^{\frac{1}{n}} = \left(\frac{ab}{0\ 0}\right)^{\frac{1}{n}}$, *a and b being positive quantities.*

Let $\sqrt[n]{\frac{a}{0}} = c$, $\sqrt[n]{\frac{b}{0}} = d$, then c , d are positive quantities;

$$\left(\frac{a}{0}\right)^{\frac{1}{n}} = c^n, \quad \left(\frac{b}{0}\right)^{\frac{1}{n}} = d^n;$$

$$\therefore \left(\frac{a}{0}\right)^{\frac{1}{n}} \times \left(\frac{b}{0}\right)^{\frac{1}{n}} = c^n \cdot d^n = (c \cdot d)^n \text{ (by Art. 46.)};$$

$$\text{Now } c^n \cdot d^n = a \cdot b,$$

$$\therefore (c \cdot d)^n = ab \text{ (by Art. 46.)},$$

$$\therefore \text{since } c, d \text{ are positive quantities } cd = \sqrt[n]{\frac{ab}{0\ 0}};$$

$$\therefore (cd)^n = \left(\frac{ab}{0\ 0}\right)^{\frac{1}{n}};$$

$$\therefore \left(\frac{a}{0}\right)^{\frac{1}{n}} \times \left(\frac{b}{0}\right)^{\frac{1}{n}} = \left(\frac{ab}{0\ 0}\right)^{\frac{1}{n}}.$$

(72.) $\frac{\left(\frac{a}{0}\right)^{\frac{1}{n}}}{\left(\frac{b}{0}\right)^{\frac{1}{n}}} = \left\{\frac{\frac{a}{0}}{\frac{b}{0}}\right\}^{\frac{1}{n}}$, *a and b being positive quantities.*

$$\text{Let } \sqrt[n]{\frac{a}{0}} = c, \quad \sqrt[n]{\frac{b}{0}} = d,$$

$$\frac{\left(\frac{a}{0}\right)^{\frac{1}{n}}}{\left(\frac{b}{0}\right)^{\frac{1}{n}}} = \frac{c^n}{d^n} = \left(\frac{c}{d}\right)^n \text{ (by Art. 47.)}$$

$$\text{And } \left(\frac{c}{d}\right)^n = \frac{a}{b},$$

∴ since c, d are positive quantities $\frac{c}{d} = \sqrt[n]{\frac{a}{b}}$,

$$\therefore \left(\frac{c}{d}\right)^m = \left\{\frac{a}{b}\right\}^{\frac{m}{n}},$$

$$\therefore \frac{\left(\frac{a}{0}\right)^{\frac{m}{n}}}{\left(\frac{b}{0}\right)^{\frac{m}{n}}} = \left\{\frac{a}{b}\right\}^{\frac{m}{n}}.$$

(73.) $\left(\frac{a}{0}\right)^{-\frac{m}{n}} \cdot \left(\frac{b}{0}\right)^{-\frac{m}{n}} = \left(\frac{a \cdot b}{0}\right)^{-\frac{m}{n}}$, a and b being positive quantities.

$$\begin{aligned} \text{For } \left(\frac{a}{0}\right)^{-\frac{m}{n}} \cdot \left(\frac{b}{0}\right)^{-\frac{m}{n}} &= \frac{1}{\left(\frac{a}{0}\right)^{\frac{m}{n}} \cdot \left(\frac{b}{0}\right)^{\frac{m}{n}}} \\ &= \frac{1}{\left(\frac{a \cdot b}{0}\right)^{\frac{m}{n}}} \quad (\text{by Art. 71.}) \\ &= \left(\frac{ab}{0}\right)^{-\frac{m}{n}}. \end{aligned}$$

(74.) $\frac{\left(\frac{a}{0}\right)^{-\frac{m}{n}}}{\left(\frac{b}{0}\right)^{-\frac{m}{n}}} = \left\{\frac{a}{b}\right\}^{-\frac{m}{n}}$, a and b being positive quantities.

$$\text{For } \frac{\left(\frac{a}{0}\right)^{-\frac{m}{n}}}{\left(\frac{b}{0}\right)^{-\frac{m}{n}}} = \frac{1}{\left(\frac{a}{0}\right)^{\frac{m}{n}} \cdot \left(\frac{b}{0}\right)^{-\frac{m}{n}}} = \frac{1}{\left\{\frac{\left(\frac{a}{0}\right)^{\frac{m}{n}}}{\left(\frac{b}{0}\right)^{\frac{m}{n}}}\right\}}$$

$$\begin{aligned}
&= \frac{1}{\left\{ \frac{a}{b} \right\}_{\frac{1}{n}}} \text{ (by Art. 72.)} \\
&= \left\{ \frac{a}{b} \right\}_{-\frac{1}{n}}
\end{aligned}$$

(75.) If $\left(a_0\right)^{\frac{m}{n}} = b$, and $\left(b_0\right)^{\frac{p}{q}} = c$, a being a positive quantity; $\left(a_0\right)^{\frac{m p}{n q}} = c$.

For let $e = \sqrt[n]{a_0}$, $f = \sqrt[q]{b_0}$, then e and f are positive quantities,

$$a = e^n \text{ and } b = f^q,$$

$$\therefore a = f^{nq} \text{ (by Art. 48.)}$$

$$\therefore \text{ since } f \text{ is a positive quantity, } f = \sqrt[nq]{a},$$

$$\therefore \left(a_0\right)^{\frac{m p}{n q}} = f^{m p};$$

$$\text{Again } b = \left(a_0\right)^{\frac{m}{n}} = e^m,$$

$$\text{and } e = f^q,$$

$$\therefore b = f^{q m} = (f^m)^q,$$

$$\therefore \text{ since } f \text{ is a positive quantity, } f^m = \sqrt[q]{b},$$

$$\therefore f^{mp} = \left(b_0\right)^{\frac{p}{q}} = c,$$

$$\therefore \left(a_0\right)^{\frac{mp}{nq}} = c.$$

(76.) If $\left(a_0\right)^{-\frac{m}{n}} = b$, and $\left(b_0\right)^{\frac{p}{q}} = c$, a being a positive quantity; $\left(a_0\right)^{-\frac{mp}{nq}} = c$.

For let $\left(a_0\right)^{\frac{n}{n}} = e$,

$$b = \left(a_0\right)^{-\frac{m}{n}} = \frac{1}{\left(a_0\right)^{\frac{m}{n}}} = \frac{1}{e},$$

$$\therefore \left(b_0\right)^{\frac{p}{q}} = \frac{\left(1_0\right)^{\frac{p}{q}}}{\left(e_0\right)^{\frac{p}{q}}} \text{ (by Art. 72.)}$$

$$= \frac{1}{\left(e_0\right)^{\frac{p}{q}}},$$

$$\therefore c = \frac{1}{\left(e_0\right)^{\frac{p}{q}}} = \frac{1}{\left(a_0\right)^{\frac{mp}{nq}}} \text{ (by Art. 75.)}$$

$$= \left(a_0\right)^{-\frac{mp}{nq}}.$$

(77.) If $\left(a_0\right)^{\frac{m}{n}} = b$, and $\left(b_0\right)^{-\frac{p}{q}} = c$, a being a positive quantity; $\left(a_0\right)^{-\frac{mp}{nq}} = c$.

For let $\left(b\right)_0^{\frac{p}{q}} = e$,

then $e = \left(a\right)_0^{\frac{mp}{nq}}$ (by Art. 75.),

$$\text{and } c = \left(b\right)_0^{-\frac{p}{q}} = \frac{1}{\left(b\right)_0^{\frac{p}{q}}} = \frac{1}{e} = \frac{1}{\left(a\right)_0^{\frac{mp}{nq}}} = \left(a\right)_0^{-\frac{mp}{nq}}.$$

(78.) If $\left(a\right)_0^{-\frac{m}{n}} = b$, and $\left(b\right)_0^{-\frac{p}{q}} = c$, a being a positive quantity; $\left(a\right)_0^{\frac{mp}{nq}} = c$.

For let $\left(b\right)_0^{\frac{p}{q}} = e$,

then $e = \left(a\right)_0^{-\frac{mp}{nq}}$ (by Art. 76.),

$$\text{and } c = \left(b\right)_0^{-\frac{p}{q}} = \frac{1}{e} = \frac{1}{\left(a\right)_0^{-\frac{mp}{nq}}} = \left(a\right)_0^{\frac{mp}{nq}}.$$

(79.) If $\left(a\right)_0^m = \left(b\right)_0^n$, $\left(a\right)_0^{mk} = \left(b\right)_0^{nk}$, where a and b are positive quantities, and m , n , k either whole or fractional, positive or negative.

For let $c = \left(a\right)_0^m = \left(b\right)_0^n$,

then c is a positive quantity,

since $c = \binom{a}{0}^m$, $(c)^k = \binom{a}{0}^{mk}$ (by Art. 75, 76, 77, 78.)

also since $c = \binom{b}{0}^n$, $(c)^k = \binom{b}{0}^{nk}$;

$$\therefore \binom{a}{0}^{mk} = \binom{b}{0}^{nk}.$$

(80.) If $\binom{a}{0}^m = \binom{b}{0}^n$, $b = \binom{a}{0}^{\frac{m}{n}}$, where a and b are positive quantities, and m and n either whole or fractional, positive or negative.

Since $\binom{b}{0}^n = \binom{a}{0}^m$,

$$\binom{b}{0}^{n \times \frac{1}{n}} = \binom{a}{0}^{m \times \frac{1}{n}} \text{ (by Art. 79.)}$$

$$\therefore \binom{b}{0}^1 = \binom{a}{0}^{\frac{m}{n}},$$

$$\text{or } b = \binom{a}{0}^{\frac{m}{n}}.$$

(81.) $\binom{1}{1}^m \times \binom{1}{1}^n = \binom{1}{1}^{m+n}$, where m and n are either whole or fractional, positive or negative.

For $\binom{1}{1}^m$ is in length = 1,

and $\binom{1}{1}^n$ is in length = 1,

$$\therefore \binom{1}{1}^m \times \binom{1}{1}^n \text{ is in length } = 1;$$

Also $\left(\frac{1}{1}\right)^{m+n}$ is in length = 1,

$\therefore \left(\frac{1}{1}\right)^m \times \left(\frac{1}{1}\right)^n$ is in length = $\left(\frac{1}{1}\right)^{m+n}$;

Also (by Art. 65.) since 1 is a positive quantity,
 $\left(\frac{1}{1}\right)^m$ is inclined to unity at an angle = $m \cdot 360^\circ$,

and $\left(\frac{1}{1}\right)^n$, at an angle = $n \cdot 360^\circ$,

therefore (by Art. 51.) $\left(\frac{1}{1}\right)^m \times \left(\frac{1}{1}\right)^n$ is inclined to
 unity at an angle = $m \cdot 360^\circ + n \cdot 360^\circ = \overline{m+n} \cdot 360^\circ$;
 Also (by Art. 65.) $\left(\frac{1}{1}\right)^{m+n}$ is inclined to unity at an
 angle = $\overline{m+n} \cdot 360^\circ$;

$\therefore \left(\frac{1}{1}\right)^m \times \left(\frac{1}{1}\right)^n$ and $\left(\frac{1}{1}\right)^{m+n}$ are inclined to unity at
 the same angle;

And they have been proved to be equal in length;

$$\therefore \left(\frac{1}{1}\right)^m \times \left(\frac{1}{1}\right)^n = \left(\frac{1}{1}\right)^{m+n}.$$

$$(82.) \quad \frac{\left(\frac{1}{1}\right)^m}{\left(\frac{1}{1}\right)^n} = \left(\frac{1}{1}\right)^{m-n}, \text{ where } m \text{ and } n \text{ are either}$$

whole or fractional, positive or negative.

$$\text{For } \frac{\left(\frac{1}{1}\right)^m}{\left(\frac{1}{1}\right)^n} = \left(\frac{1}{1}\right)^m \cdot \left(\frac{1}{1}\right)^{-n} = \left(\frac{1}{1}\right)^{m-n} \text{ (by Art. 81).}$$

(83.) $\left(\frac{1}{p}\right)^m = \left(\frac{1}{1}\right)^{mp}$, where m is either whole or fractional, positive or negative.

For (by Art. 65.) $\left(\frac{1}{p}\right)^m$ is inclined to unity at an angle $= mp \cdot 360^\circ$;

And (by same Art.) $\left(\frac{1}{1}\right)^{mp}$ is inclined to unity at the same angle,

$$\therefore \left(\frac{1}{p}\right)^m = \left(\frac{1}{1}\right)^{mp}.$$

(84.) $\left(\frac{a}{p}\right)^m = \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{p}\right)^m$, where m is either whole or fractional, positive or negative.

For $\left(\frac{1}{p}\right)^m$ is in length $= 1$,

$$\therefore \left(\frac{a}{p}\right)^m \cdot \left(\frac{1}{p}\right)^m \text{ is in length } = \left(\frac{a}{0}\right)^m;$$

And $\left(\frac{a}{p}\right)^m$ is in length $= \left(\frac{a}{0}\right)^m$;

$$\therefore \left(\frac{a}{p}\right)^m \text{ is in length } = \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{p}\right)^m;$$

Let a be inclined to unity at an angle $= A$,
 A being positive and less than 360° ,

then $\left(\frac{a}{0}\right)^m$ is inclined to unity at an angle $= mA$,

and $\left(\frac{1}{p}\right)^m \dots\dots\dots = mp \cdot 360^\circ$;

$$\therefore \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{p}\right)^m \dots\dots\dots = mA + mp \cdot 360^\circ = m(A + p \cdot 360^\circ);$$

Also $\left(\frac{a}{p}\right)^m \dots\dots\dots = m(A + p \cdot 360^\circ)$;

$$\therefore \left(\frac{a}{p}\right)^m = \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{p}\right)^m.$$

(85.) *Any quantity a may be expressed in the form $b \left(\frac{1}{1} \right)^n$, where b is a positive quantity, and n positive and less than unity.*

For let b be in length $= a$, and let a be inclined to unity at an angle $= A$, A being positive and less than 360° ,

Take $n = \frac{A}{360^\circ}$, then n is positive and less than unity,

And (by Art. 65.) $\left(\frac{1}{1} \right)^n$ is inclined to unity at an angle $= n \cdot 360^\circ$

$$= \frac{A}{360^\circ} \cdot 360^\circ = A;$$

$\therefore b \left(\frac{1}{1} \right)^n$ is inclined to unity at an angle $= A$, that is, at the same angle that a is inclined to unity;

And $b \left(\frac{1}{1} \right)^n$ is in length equal to a ;

$$\therefore b \left(\frac{1}{1} \right)^n = a.$$

(86.) *If $a = b \left(\frac{1}{1} \right)^n$, where b is a positive quantity, and n positive and less than 1; $\left(\frac{a}{b} \right)^m = \left(\frac{b}{b} \right)^m \cdot \left(\frac{1}{1} \right)^{mn}$, where m may be either whole or fractional, positive or negative.*

For let a be inclined to unity at an angle $= A$, A being positive and less than 360° ,

Then (by preceding Article)

$$\frac{A}{360^\circ} = n, \text{ or } A = n \cdot 360^\circ;$$

$\left(\frac{a}{0}\right)^m$ is inclined to unity at an angle $= mA$;

And $\left(\frac{1}{1}\right)^{mn} \dots \dots \dots = mn \cdot 360 = mA$,

\therefore since b is a positive quantity,

$\left(\frac{b}{0}\right)^m \left(\frac{1}{1}\right)^{mn}$ is inclined to unity at an angle $= mA$;

And since b is in length $= a$,

$\left(\frac{b}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{mn}$ is in length $= \left(\frac{a}{0}\right)^m$;

$$\therefore \left(\frac{b}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{mn} = \left(\frac{a}{0}\right)^m.$$

(87.) If $a = b \left(\frac{1}{1}\right)^n$, where b is a positive quantity, and n positive and less than unity; $\left(\frac{a}{p}\right)^m = \left(\frac{b}{p}\right)^m \cdot \left(\frac{1}{1}\right)^{mn}$, where m may be either whole or fractional, positive or negative.

$$\begin{aligned} \text{For } \left(\frac{a}{p}\right)^m &= \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{p}\right)^m \text{ (by Art. 84.)} \\ &= \left(\frac{b}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{mn} \cdot \left(\frac{1}{p}\right)^m \text{ (by Art. 86.)} \\ &= \left(\frac{b}{0}\right)^m \cdot \left(\frac{1}{p}\right)^m \cdot \left(\frac{1}{1}\right)^{mn} \\ &= \left(\frac{b}{p}\right)^m \cdot \left(\frac{1}{1}\right)^{mn}. \end{aligned}$$

(88.) $\left(\frac{a}{p}\right)^m \cdot \left(\frac{a}{p}\right)^n = \left(\frac{a}{p}\right)^{m+n}$, where m, n may be either whole or fractional, positive or negative.

For let $a = b \left(\frac{1}{1}\right)^k$, where b is a positive quantity, and k positive and less than 1,

$$\begin{aligned} \text{then } \left(\frac{a}{p}\right)^m &= \left(\frac{b}{p}\right)^m \cdot \left(\frac{1}{1}\right)^{mk} \\ &= \left(\frac{b}{0}\right)^m \cdot \left(\frac{1}{p}\right)^m \cdot \left(\frac{1}{1}\right)^{mk} \\ &= \left(\frac{b}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{mp} \cdot \left(\frac{1}{1}\right)^{mk}; \end{aligned}$$

$$\text{and } \left(\frac{a}{p}\right)^n = \left(\frac{b}{0}\right)^n \cdot \left(\frac{1}{1}\right)^{np} \cdot \left(\frac{1}{1}\right)^{nk};$$

$$\begin{aligned} \therefore \left(\frac{a}{p}\right)^m \cdot \left(\frac{a}{p}\right)^n &= \left(\frac{b}{0}\right)^m \cdot \left(\frac{b}{0}\right)^n \cdot \left(\frac{1}{1}\right)^{mp} \cdot \left(\frac{1}{1}\right)^{np} \cdot \left(\frac{1}{1}\right)^{mk} \cdot \left(\frac{1}{1}\right)^{nk} \\ &= \left(\frac{b}{0}\right)^{m+n} \cdot \left(\frac{1}{1}\right)^{\overline{m+n \cdot p}} \cdot \left(\frac{1}{1}\right)^{\overline{m+n \cdot k}} \text{ (Art. 67, 68, 69, 81.)} \\ &= \left(\frac{b}{p}\right)^{m+n} \cdot \left(\frac{1}{1}\right)^{\overline{m+n \cdot k}} \\ &= \left(\frac{a}{p}\right)^{m+n}. \end{aligned}$$

$$(89.) \quad \frac{\left(\frac{a}{p}\right)^m}{\left(\frac{a}{p}\right)^n} = \left(\frac{a}{p}\right)^{m-n}, \text{ where } m \text{ and } n \text{ are either}$$

whole or fractional, positive or negative.

$$\text{For } \frac{\left(\frac{a}{p}\right)^m}{\left(\frac{a}{p}\right)^n} = \left(\frac{a}{p}\right)^m \cdot \left(\frac{a}{p}\right)^{-n} = \left(\frac{a}{p}\right)^{m-n} \text{ (by Art. 88).}$$

$$(90.) \quad \left(\frac{a}{p}\right)^0 = 1.$$

$$\text{For } \left(\frac{a}{p}\right)^{m-n} = \frac{\left(\frac{a}{p}\right)^m}{\left(\frac{a}{p}\right)^n},$$

$$\text{Let } m = n,$$

$$\left(\frac{a}{p}\right)^0 = 1.$$

(91.) *If $a \cdot b = c$, and a be inclined to unity at an angle = A , and b at an angle = B ; A and B being positive and less than 360° ;*

$$\begin{aligned} \left(\frac{a}{0}\right)^m \cdot \left(\frac{b}{0}\right)^m &= \left(\frac{c}{0}\right)^m, \text{ if } A + B \text{ be less than } 360^\circ, \\ &= \left(\frac{c}{1}\right)^m, \text{ if } A + B \text{ be greater than } 360^\circ, \end{aligned}$$

where m may be either whole or fractional, positive or negative.

For let a be in length = e , and b be in length = f ; e and f being positive quantities;

$$\therefore c \text{ is in length} = e \cdot f;$$

$$\therefore \left(\frac{c}{0}\right)^m \text{ or } \left(\frac{c}{1}\right)^m \text{ is in length} = \left(\frac{e \cdot f}{0}\right)^m;$$

$$\begin{aligned} \text{And } \left(\frac{a}{0}\right)^m \cdot \left(\frac{b}{0}\right)^m &\text{ is in length} = (e)^m \cdot (f)^m \\ &= \left(\frac{e \cdot f}{0}\right)^m \text{ (by Art. 71, 73);} \end{aligned}$$

$$\therefore \left(\frac{a}{0}\right)^m \cdot \left(\frac{b}{0}\right)^m \text{ is in length equal to } \left(\frac{c}{0}\right)^m \text{ or } \left(\frac{c}{1}\right)^m;$$

Also $\left(a\right)_0^m$ is inclined to unity at an angle $= mA$,

$$\left(b\right)_0^m \dots\dots\dots = mB;$$

$$\therefore \left(a\right)_0^m \cdot \left(b\right)_0^m \dots\dots\dots = mA + mB \\ = m.(A + B);$$

Again $\because c = a \cdot b$, c is inclined to unity at an angle $= A + B$,

\therefore if $A + B$ be less than 360° , c represents c considered as inclined to unity at the angle $= A + B$;

$\therefore \left(c\right)_0^m$ is inclined to unity at an angle $= m(A + B)$;

\therefore when $A + B$ is less than 360° , $\left(a\right)_0^m \cdot \left(b\right)_0^m = \left(c\right)_0^m$;

But if $A + B$ be greater than 360° ; since $A + B$ is less than 2.360° , c represents c considered as inclined to unity at the angle $= A + B$;

$\therefore \left(c\right)_1^m$ is inclined to unity at an angle $= m.(A + B)$;

\therefore in this case $\left(a\right)_0^m \cdot \left(b\right)_0^m = \left(c\right)_1^m$.

(92.) *If $a \cdot b = c$, and a be inclined to unity at an angle $= A$, and b at an angle $= B$, A and B being positive and less than 360° ;*

$$\left(\frac{a}{p}\right)^m \cdot \left(\frac{b}{q}\right)^m = \left(\frac{c}{p+q}\right)^m, \text{ if } A + B \text{ be less than } 360^\circ,$$

$$= \left(\frac{c}{p+q+1}\right)^m, \dots\dots\dots \text{greater} \dots\dots\dots,$$

where m may be either whole or fractional, positive or negative.

$$\text{For } \left(\frac{a}{p}\right)^m = \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{p}\right)^m = \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{mp},$$

$$\left(\frac{b}{q}\right)^m = \dots\dots\dots \left(\frac{b}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{mq};$$

$$\therefore \left(\frac{a}{p}\right)^m \cdot \left(\frac{b}{q}\right)^m = \left(\frac{a}{0}\right)^m \cdot \left(\frac{b}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{m \cdot \overline{p+q}}$$

$$\begin{aligned} &= \left(\frac{c}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{m \cdot \overline{p+q}}, \text{ if } A + B \text{ be less than } 360^\circ, \\ &= \left(\frac{c}{1}\right)^m \cdot \left(\frac{1}{1}\right)^{m \cdot \overline{p+q}}, \dots\dots\dots \text{greater} \dots\dots\dots \end{aligned} \left. \vphantom{\begin{aligned} &= \left(\frac{c}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{m \cdot \overline{p+q}}, \text{ if } A + B \text{ be less than } 360^\circ, \\ &= \left(\frac{c}{1}\right)^m \cdot \left(\frac{1}{1}\right)^{m \cdot \overline{p+q}}, \dots\dots\dots \text{greater} \dots\dots\dots \end{aligned}} \right\} \text{by Art. 91.}$$

$$= \left(\frac{c}{p+q}\right)^m, \text{ if } A + B \text{ be less than } 360^\circ,$$

$$= \left(\frac{c}{p+q+1}\right)^m, \dots\dots\dots \text{greater} \dots\dots\dots$$

(93.) If $\frac{a}{b} = c$, and a be inclined to unity at

an angle = A , and b at an angle = B ; A and B being positive and less than 360° ;

$$\frac{\left(\frac{a}{b}\right)^m}{\left(\frac{b}{0}\right)^m} = \left(\frac{c}{0}\right)^m, \text{ if } A \text{ be greater than } B$$

$$= \left(\frac{c}{-1}\right)^m \dots\dots\dots \text{less} \dots\dots\dots$$

where m may be either whole or fractional, positive or negative.

$$\text{For since } \frac{a}{b} = c, b \cdot c = a,$$

$$\therefore \left(\frac{b}{0}\right)^m \cdot \left(\frac{c}{0}\right)^m \text{ is equal in length to } \left(\frac{a}{0}\right)^m,$$

$$\therefore \frac{\left(\frac{a}{0}\right)^m}{\left(\frac{b}{0}\right)^m} \text{ is equal in length to } \left(\frac{c}{0}\right)^m \text{ or } \left(\frac{c}{-1}\right)^m;$$

$$\text{Now } \left(\frac{a}{0}\right)^m \text{ is inclined to unity at an angle} = mA,$$

$$\left(\frac{b}{0}\right)^m \dots\dots\dots = mB,$$

$$\therefore \frac{\left(\frac{a}{0}\right)^m}{\left(\frac{b}{0}\right)^m} \dots\dots\dots = mA - mB$$

$$= m(A - B);$$

$$\text{Again } \because c = \frac{a}{b}, c \text{ is inclined to unity at an angle} = A - B,$$

And if A be greater than B , $A - B$ is positive; and it is also less than 360° , since A, B are each less than 360° ;

therefore in this case c_0 represents c considered as inclined to unity at the angle $= A - B$;

$\therefore (c_0)^m$ is inclined to unity at an angle $= m(A - B)$;

\therefore when A is greater than B , $\frac{(a_0)^m}{(b_0)^m} = (c_0)^m$.

But if A be less than B , $A - B$ is negative ;
therefore in this case c_{-1} represents c considered as inclined to unity at the angle $A - B$;

$\therefore (c_{-1})^m$ is inclined to unity at an angle $= m(A - B)$;

\therefore when A is less than B , $\frac{(a_0)^m}{(b_0)^m} = (c_{-1})^m$.

(94.) COR. Hence if $\frac{1}{b} = c$,

$\frac{1}{(b_0)^m} = (c_0)^m$, if b be a positive quantity,

$= (c_{-1})^m$, if b be not a positive quantity.

For putting $a = 1$, in the preceding Article, $A = 0$,

and if b be a positive quantity, $B = 0$;

$\therefore A - B = 0$;

$$\therefore \frac{\binom{1}{0}^m}{\binom{b}{0}^m} = \binom{c}{0}^m, \text{ or } \frac{1}{\binom{b}{0}^m} = \binom{c}{0}^m;$$

But if b be not a positive quantity, $A - B = 0 - B$,
 $\therefore A - B$ is negative;

$$\therefore \frac{1}{\binom{b}{0}^m} = \binom{c}{-1}^m.$$

(95.) If $\frac{a}{b} = c$, and a be inclined to unity at
 an angle = A , and b , at an angle = B ; A , B
 being positive and less than 360° ;

$$\frac{\binom{a}{p}^m}{\binom{b}{q}^m} = \binom{c}{p-q}^m, \text{ if } A \text{ be greater than } B,$$

$$= \binom{c}{p-q-1}^m, \dots \dots \dots \text{less} \dots \dots \dots,$$

where m may be either whole or fractional, positive
 or negative.

$$\text{For } \frac{\binom{a}{p}^m}{\binom{b}{q}^m} = \frac{\binom{a}{0}^m \cdot \binom{1}{1}^{mp}}{\binom{b}{0}^m \cdot \binom{1}{1}^{mq}} = \frac{\binom{a}{0}^m}{\binom{b}{0}^m} \cdot \binom{1}{1}^{m \cdot p - q}$$

$$\left. \begin{aligned} &= \binom{c}{0}^m \cdot \binom{1}{1}^{m \cdot p - q}, \text{ if } A \text{ be greater than } B, \\ &= \binom{c}{-1}^m \cdot \binom{1}{1}^{m \cdot p - q}, \dots \dots \dots \text{less} \dots \dots \dots, \end{aligned} \right\} \text{ (by Art. 93.)}$$

$$= \binom{c}{p-q}^m, \text{ if } A \text{ be greater than } B,$$

$$= \binom{c}{p-q-1}^m, \dots \dots \dots \text{less} \dots \dots \dots$$

(96.) COR. Hence, if $\frac{1}{b} = c$,

$$\frac{1}{\binom{b}{q}^m} = \binom{c}{-q}^m, \text{ if } b \text{ be a positive quantity,}$$

$$= \binom{c}{-q-1}^m, \text{ if } b \text{ be not a positive quantity.}$$

(97.) $\left(\frac{a}{p}\right)^{\frac{m}{n}} = \left(\frac{a}{p}\right)^{\frac{km}{kn}}$, where m, n, k are whole numbers.

For let c be a positive quantity equal in length to a ;

$$\text{then } \left(\frac{c}{0}\right)^{\frac{m}{n}} \text{ is in length } = \left(\frac{a}{p}\right)^{\frac{m}{n}} \text{ (by Art. 59.)}$$

$$\text{and } \left(\frac{c}{0}\right)^{\frac{km}{kn}} \dots \dots \dots = \left(\frac{a}{p}\right)^{\frac{km}{kn}} \dots \dots \dots$$

$$\text{But } \left(\frac{c}{0}\right)^{\frac{m}{n}} = \left(\frac{c}{0}\right)^{\frac{km}{kn}} \text{ (by Art. 66),}$$

$$\therefore \left(\frac{a}{p}\right)^{\frac{m}{n}} \text{ is in length } = \left(\frac{a}{p}\right)^{\frac{km}{kn}};$$

Let a be inclined to unity at an angle $= A$,
 A being positive and less than 360° ,

then $\left(a_p\right)^{\frac{m}{n}}$ is inclined to unity at an angle $= \frac{m}{n}(A+p.360^\circ)$,

and $\left(a_p\right)^{\frac{km}{kn}} \dots \dots \dots = \frac{km}{kn}(A+p.360^\circ)$

$$= \frac{m}{n}(A+p.360^\circ);$$

$$\therefore \left(a_p\right)^{\frac{m}{n}} = \left(a_p\right)^{\frac{km}{kn}}.$$

(98.) *If $a^m = b$, and $b^n = c$; c is in length $= a^{mn}$; where m, n may be either whole or fractional, positive or negative.*

For let e be a positive quantity in length $= a$;

then $\left(e_0\right)^m$ is in length $= a^m = b$;

$\therefore c$ is in length $= \left(e_0\right)^{mn}$ (by Art 75, 76, 77, 78.)

But $\left(e_0\right)^{mn}$ is in length $= a^{mn}$,

since e is length $= a$;

$\therefore c$ is in length $= a^{mn}$.

(99.) *Let $\left(a_p\right)^{\frac{k}{l}} = b$, and $\left(b_q\right)^{\frac{m}{n}} = c$, where $\frac{k}{l}, \frac{m}{n}$ are fractions in their lowest terms; to investigate in what cases c is a value of $a^{\frac{km}{ln}}$.*

(By Art. 98.) c is in length $= a^{\frac{km}{ln}}$;

therefore if any value of $a^{\frac{km}{ln}}$ be inclined to unity at the same angle that c is, that value of $a^{\frac{km}{ln}}$ is equal to c ;

Assume $(a_x)^{\frac{km}{ln}}$ to be such a value of $a^{\frac{km}{ln}}$,

and let a be inclined to unity at an angle $= A$,
 A being positive and less than 360° ,

then $(a_x)^{\frac{km}{ln}}$ is inclined to unity at an angle

$$= \frac{km}{ln} (A + x \cdot 360^\circ);$$

Also since $(a_p)^{\frac{k}{l}} = b$, b is inclined to unity at an angle

$$= \frac{k}{l} (A + p \cdot 360^\circ);$$

Let $\frac{k}{l} (A + p \cdot 360^\circ) = B + r \cdot 360^\circ$, where B is positive and less than 360° , and $r = 0$, or some whole number, either positive, or negative,

$\therefore (b_q)^{\frac{m}{n}} = c$, c is inclined to unity at an angle

$$= \frac{m}{n} (B + q \cdot 360^\circ),$$

$$\text{But } B = \frac{k}{l} (A + p \cdot 360^\circ) - r \cdot 360^\circ,$$

$\therefore c$ is inclined to unity at an angle

$$= \frac{km}{ln} (A + p \cdot 360^\circ) + \frac{m}{n} \cdot (q - r) 360^\circ;$$

But $(a_x)^{\frac{km}{ln}}$ is inclined to unity at an angle

$$= \frac{km}{ln} (A + x \cdot 360^\circ);$$

\therefore since $(a_x)^{\frac{km}{ln}}$ is inclined to unity at the same angle that c is,

$$\begin{aligned} & \frac{km}{ln} (A + x \cdot 360^\circ) + y \cdot 360^\circ \\ &= \frac{km}{ln} \cdot (A + p \cdot 360^\circ) + \frac{m}{n} (q - r) 360^\circ, \end{aligned}$$

where y must either $= 0$, or some whole number, either positive, or negative;

$$\begin{aligned} \therefore & \frac{km}{ln} \cdot x \cdot 360^\circ + y \cdot 360^\circ \\ &= \frac{km}{ln} \cdot p \cdot 360^\circ + \frac{m}{n} \cdot (q - r) 360^\circ, \end{aligned}$$

$$\therefore kmx + lny = kmp + lm(q-r),$$

$$\therefore kx + ln\frac{y}{m} = kp + l(q-r);$$

In which equation, that x may be a whole number, it is necessary that y should either $= 0$, or some multiple of m ; let $y = mz$,

$$\text{then } kx + lnz = kp + l(q-r),$$

where if k be prime to ln , integer values of x and z can always be found, which will satisfy the conditions of the equation,

but by the hypothesis k is prime to l ,

\therefore whenever k is prime to n , k is also prime to ln ,

\therefore whenever k is prime to n , c is a value of $a^{\frac{km}{ln}}$.

If k be not prime to n ,

let e be the greatest common measure of k and n ,

and let $k = ef$, $n = eg$,

then $efx + legz = efp + l(q-r)$,

$$\therefore fx + lgz = fp + \frac{l(q-r)}{e};$$

the conditions of which equation cannot be satisfied by integer values of x and z , unless $\frac{l(q-r)}{e} = 0$, or some whole number, either positive, or negative;

And $\frac{l(q-r)}{e}$ cannot $= 0$, unless $q - r = 0$,

Also since l is prime to k , l is prime to e a part of k ,

$\therefore \frac{l(q-r)}{e}$ cannot $=$ a whole number, unless $q - r$ be a multiple of e ;

\therefore if k be not prime to n , c is not a value of $a^{\frac{km}{tn}}$, unless, either $q - r = 0$, or the greatest common measure of k and n be also a measure of $q - r$.

(100.) COR. 1. It will appear from a similar investigation, if $(a_p)^{-\frac{k}{t}} = b$, and $(b_q)^{-\frac{m}{n}} = c$, that c is a value of $a^{\frac{km}{tn}}$, whenever k is prime to n ; and that when k is not prime to n , c is not a value of $a^{\frac{km}{tn}}$, unless either $q - r = 0$, or the greatest common measure of k and n be also a measure of $q - r$; also that if $(a_p)^{-\frac{k}{t}} = b$, and $(b_q)^{\frac{m}{n}} = c$, or $(a_p)^{\frac{k}{t}} = b$, and $(b_q)^{-\frac{m}{n}} = c$, c is a value of $a^{-\frac{km}{tn}}$ in the same cases.

(101.) COR. 2. Let $(a)_p^t = b$, and let a be inclined to unity at an angle $= A$, A being positive and less than 360° , and let $B + q \cdot 360^\circ = s(A + p \cdot 360^\circ)$, where B is positive and less than 360° , and $q = 0$ or some whole number either positive or negative, and let $(b)_q^s = c$, and let s and t be either whole or fractional, positive or negative; then $(a)_p^{st} = c$.

First let s and t be both positive whole numbers,

then $c = a^{st}$ (by Art. 48),

also a^{st} has no other value different from c (by Art. 43.)

$$\therefore (a)_p^{st} = c.$$

Next let s and t be both positive, but one or both fractions.

$$\text{Let } s = \frac{k}{l}, \quad t = \frac{m}{n},$$

$$\text{then } B + q \cdot 360^\circ = \frac{k}{l} \cdot (A + p \cdot 360^\circ);$$

$$\text{But } B + r \cdot 360^\circ = \frac{k}{l} (A + p \cdot 360^\circ) \text{ (by Art. 99),}$$

$$\therefore q = r, \text{ or } q - r = 0;$$

\therefore the equation $kx + lnx = kp + l(q-r)$ (in Art. 99)

becomes $kx + lnx = kp$;

II

where if we make $z = 0$, and $x = p$, the conditions of the equation are answered;

$$\therefore c = \left(a\right)_p^{\frac{km}{t}} = \left(a\right)_p^{st}.$$

Next let s be negative and t positive, or s positive and t negative, or both s and t negative.

These cases may be demonstrated by means of Art. 100, in the same manner that the preceding case was demonstrated by means of Art. 99; therefore whether s and t be whole or fractional, positive or negative,

$$c = \left(a\right)_p^{st}.$$

(102.) *If $\left(a\right)_p^m = b$, where m is either whole or fractional, positive or negative, and a be inclined to unity at an angle $= A$, A being positive and less than 360° , and $m \cdot (A + p \cdot 360^\circ) = B + q \cdot 360^\circ$, where B is positive and less than 360° , and $q = 0$, or some whole number, either positive or negative;*

$$\text{then } \left(b\right)_q^{\frac{1}{m}} = a.$$

$$\text{For let } \left(b\right)_q^{\frac{1}{m}} = c,$$

then $c = (a)^{\frac{m}{n}}$ (by Art. 101.)

$$= (a)^1 = a;$$

$$\therefore (b)^{\frac{1}{n}} = a.$$

(103.) *If c be a value of $a^{\frac{k}{l}}$ and also a value of $b^{\frac{m}{n}}$, where $\frac{k}{l}$, $\frac{m}{n}$ are fractions in their lowest terms; then if k be prime to m , b is a value of $a^{\frac{kn}{lm}}$.*

For since c is a value of $b^{\frac{m}{n}}$, b is a value of $c^{\frac{n}{m}}$ (by Art. 102.)

\therefore since k is prime m , b is a value of $a^{\frac{kn}{lm}}$ (by Art. 99).

(104.) *COR.* In like manner it may be proved, if c be a value of $a^{-\frac{k}{l}}$ and also a value of $b^{\frac{m}{n}}$, where $\frac{k}{l}$, $\frac{m}{n}$ are fractions in their lowest terms; that, if k be prime to m , b is a value of $a^{-\frac{kn}{lm}}$.

(105.) *The values of the square root of -1 are inclined to unity at angles $= 90^\circ$ and 270° .*

For (by Art. 61.) $\sqrt[n]{-1}$ has two values,

viz. $(-1)_0^{\frac{1}{2}}$, and $(-1)_1^{\frac{1}{2}}$;

Now -1 is inclined to unity at an angle $= 180^\circ$,

$$\therefore (-1)_0^{\frac{1}{2}} \dots \dots \dots = \frac{1}{2}(180^\circ) = 90^\circ,$$

$$\text{and } (-1)_1^{\frac{1}{2}} \dots \dots \dots = \frac{1}{2}(180^\circ + 360^\circ) \\ = 270^\circ.$$

(106.) If $(-1)_0^{\frac{1}{2}}$ be represented by $+\sqrt{-1}$,
 $(-1)_1^{\frac{1}{2}}$ will be represented by $-\sqrt{-1}$.

For by the preceding Article

$$(-1)_0^{\frac{1}{2}} \text{ is inclined to unity at an angle } = 90^\circ,$$

$$(-1)_1^{\frac{1}{2}} \dots \dots \dots = 270^\circ,$$

$$\therefore (-1)_1^{\frac{1}{2}} \text{ is inclined to } (-1)_0^{\frac{1}{2}} \text{ at an angle} \\ = 270^\circ - 90^\circ = 180^\circ;$$

$$\therefore \text{ since } (-1)_0^{\frac{1}{2}} \text{ is represented by } +\sqrt{-1},$$

$$(-1)_1^{\frac{1}{2}} \text{ will be represented by } -\sqrt{-1} \text{ (by Art. 8).}$$

(107.) The values of $1^{\frac{1}{2}}$ are $1, +\sqrt{-1}, -1, -\sqrt{-1}$.

$$\text{For } (1)_0^{\frac{1}{2}} = 1,$$

$$\left(\frac{1}{1}\right)^{\frac{1}{4}} \text{ is inclined to unity at an angle } = \frac{360^\circ}{4} = 90^\circ,$$

$$\left(\frac{1}{2}\right)^{\frac{1}{4}} \dots\dots\dots = \frac{2 \cdot 360^\circ}{4} = 180^\circ,$$

$$\left(\frac{1}{3}\right)^{\frac{1}{4}} \dots\dots\dots = \frac{3 \cdot 360^\circ}{4} = 270^\circ;$$

$$\text{But } \sqrt{-1} \dots\dots\dots = 90^\circ,$$

$$-1 \dots\dots\dots = 180^\circ,$$

$$-\sqrt{-1} \dots\dots\dots = 270^\circ;$$

$$\therefore \left(\frac{1}{0}\right)^{\frac{1}{4}} = 1, \left(\frac{1}{1}\right)^{\frac{1}{4}} = \sqrt{-1}, \left(\frac{1}{2}\right)^{\frac{1}{4}} = -1, \left(\frac{1}{3}\right)^{\frac{1}{4}} = -\sqrt{-1}.$$

$$(108.) \quad \left(\frac{1}{1}\right)^{-\frac{1}{4}} = -\sqrt{-1}.$$

For (by Art. 65.)

$$\left(\frac{1}{1}\right)^{-\frac{1}{4}} \text{ is inclined to unity at an angle } = -\frac{1}{4} 360^\circ,$$

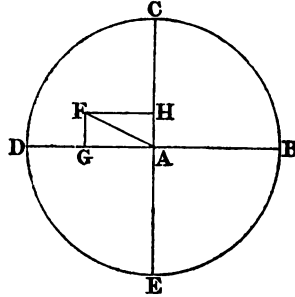
$$\text{and } \left(\frac{1}{3}\right)^{\frac{1}{4}} \dots\dots\dots = \frac{3}{4} 360^\circ$$

$$\dots\dots\dots = 360^\circ - \frac{1}{4} 360^\circ,$$

$$\therefore \left(\frac{1}{1}\right)^{-\frac{1}{4}} \text{ coincides with } \left(\frac{1}{3}\right)^{\frac{1}{4}},$$

$$\therefore \left(\frac{1}{1}\right)^{-\frac{1}{4}} = -\sqrt{-1} \text{ (by Art. 107).}$$

(109.) *Any quantity may be expressed in the form $\pm a \pm b\sqrt{-1}$, where a, b are positive quantities.*



Let A be the origin, and $AB = \text{unity}$;

From centre A with radius AB describe circle $BCDE$, and produce BA to D , and draw EAC perpendicular to DB ;

Let BC be the direction in which positive angles are measured,

then $AC = \sqrt{-1}$, $AD = -1$, $AE = -\sqrt{-1}$;

Let c be the given quantity;

Draw $AF = c$, and draw FG perpendicular to DA , and FH perpendicular to AC ;

Since $AGFH$ is a parallelogram,

$$AF = AG + AH \text{ (by Art. 3.)}$$

$$\text{or } c = AG + AH;$$

Let a be a positive quantity in length = AG ,

b = AH ;

Then $AB : a :: AD : AG$,

or $1 : a :: -1 : AG$,

$\therefore AG = a \times -1 = -a$;

And $AB : b :: AC : AH$,

or $1 : b :: \sqrt{-1} : AH$,

$\therefore AH = b\sqrt{-1}$;

$\therefore c = -a + b\sqrt{-1}$;

In like manner it may be proved in any other case.

(110.) COR. 1. Let c be inclined to unity at an angle = C , C being positive and less than 360° ;

Then if C be less than 90° ,

c is of the form $+a + b\sqrt{-1}$;

if C be greater than 90° and less than 180° ,

c is of the form $-a + b\sqrt{-1}$;

if C be greater than 180° and less than 270° ,

c is of the form $-a - b\sqrt{-1}$;

if C be greater than 270° ,

c is of the form $+a - b\sqrt{-1}$;

(111.) COR. 2. Hence if $c = e \left(\frac{1}{1} \right)^n$, e being a positive quantity, and n positive and less than 1;

Then if n be less than $\frac{1}{4}$,

c is of the form $+ a + b \sqrt{-1}$;

if n be greater than $\frac{1}{4}$ and less than $\frac{1}{2}$,

c is of the form $- a + b \sqrt{-1}$;

if n be greater than $\frac{1}{2}$ and less than $\frac{3}{4}$,

c is of the form $- a - b \sqrt{-1}$;

if n be greater than $\frac{3}{4}$,

c is of the form $+ a - b \sqrt{-1}$.

(112.) COR. 3. When n is less than $\frac{1}{4}$,

as n increases, $\frac{b}{a}$ increases; and conversely;

when n is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$,

as n increases, $\frac{b}{a}$ decreases; and conversely;

when n is greater than $\frac{1}{2}$ and less than $\frac{3}{4}$,

as n increases, $\frac{b}{a}$ increases; and conversely;

when n is greater than $\frac{3}{4}$,

as n increases, $\frac{b}{a}$ decreases; and conversely.

(113.) To express the values of $1^{\frac{1}{6}}$ in the form $\pm a \pm b\sqrt{-1}$, where a, b are positive quantities.

The values of

$$1^{\frac{1}{6}} \text{ are } \binom{1}{0}^{\frac{1}{6}}, \binom{1}{1}^{\frac{1}{6}}, \binom{1}{2}^{\frac{1}{6}}, \binom{1}{3}^{\frac{1}{6}}, \binom{1}{4}^{\frac{1}{6}}, \binom{1}{5}^{\frac{1}{6}};$$

$$\text{of which } \binom{1}{0}^{\frac{1}{6}} = 1;$$

And since $\frac{1}{6}$ is less than $\frac{1}{4}$,

$$\binom{1}{1}^{\frac{1}{6}} \text{ is of the form } a + b\sqrt{-1};$$

$$\binom{1}{2}^{\frac{1}{6}} = \binom{1}{1}^{\frac{2}{6}} = \binom{1}{1}^{\frac{1}{3}},$$

\therefore since $\frac{1}{3}$ is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$,

$$\binom{1}{2}^{\frac{1}{6}} \text{ is of the form } -a + b\sqrt{-1};$$

$$\binom{1}{3}^{\frac{1}{6}} = \binom{1}{1}^{\frac{3}{6}} = \binom{1}{1}^{\frac{1}{2}} = -1;$$

$$\binom{1}{4}^{\frac{1}{6}} = \binom{1}{1}^{\frac{4}{6}} = \binom{1}{1}^{\frac{2}{3}},$$

\therefore since $\frac{2}{3}$ is greater than $\frac{1}{2}$ and less than $\frac{3}{4}$,

$$\binom{1}{4}^{\frac{1}{6}} \text{ is of the form } -a, -b\sqrt{-1};$$

$$\binom{1}{5}^{\frac{1}{6}} = \binom{1}{1}^{\frac{5}{6}},$$

\therefore since $\frac{5}{6}$ is greater than $\frac{3}{4}$,

$\left(1\right)_s^{\frac{1}{8}}$ is of the form $+a - b\sqrt{-1}$;

Let $1^{\frac{11}{8}} = x$,

then $x^6 = 1$, or $x^6 - 1 = 0$,

From the solution of this equation we obtain the following values of x ;

$$1, -\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}, -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},$$

$$-1, \frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}, \frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1};$$

$$\therefore \left(1\right)_0^{\frac{1}{8}} = 1,$$

$$\left(1\right)_1^{\frac{1}{8}} = \frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},$$

$$\left(1\right)_2^{\frac{1}{8}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},$$

$$\left(1\right)_3^{\frac{1}{8}} = -1,$$

$$\left(1\right)_4^{\frac{1}{8}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1},$$

$$\left(1\right)_5^{\frac{1}{8}} = \frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}.$$

(114.) If $a + b \binom{1}{1}^m + c \binom{1}{1}^n = f \binom{1}{1}^p$, and a, b, c, f be positive quantities,

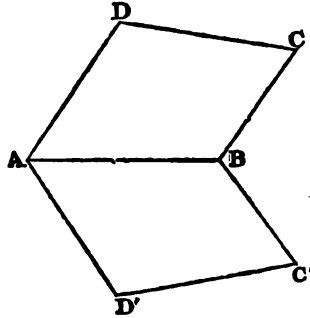
$$a + b \binom{1}{-1}^m + c \binom{1}{-1}^n = f \binom{1}{-1}^p.$$

Let A be the origin, and $AB = a$.

From B draw BC parallel and equal in length to the line which represents $b \binom{1}{1}^m$, and from C draw CD parallel and equal in length to the line which represents $c \binom{1}{1}^n$, and join AD ;

$$\begin{aligned} AD &= a + b \binom{1}{1}^m + c \binom{1}{1}^n \text{ (by Art. 7.)} \\ &= f \binom{1}{1}^p; \end{aligned}$$

On the other side of AB draw BC' making angle $ABC' = ABC$ and make BC' in length = BC ,



In like manner make angle $BC'D' = \text{angle } BCD$, and $C'D'$ in length = CD ,

Join AD' ;

Since BC' is equal in length to BC , and angle ABC' equal to ABC , but in the opposite direction, and BC parallel and equal in length to the line which represents $b \binom{1}{1}^m$; BC' will be parallel and equal in length to the line which represents $b \binom{1}{-1}^m$;

In like manner $C'D'$ is parallel and equal in length to the line which represents $c \binom{1}{-1}^n$;

$$\therefore AD' = a + b \binom{1}{-1}^m + c \binom{1}{-1}^n;$$

But since figures $ABCD$, $ABC'D'$ are similar and equal, AD' is equal in length to AD , and angle BAD' equal to BAD , but in the opposite direction,

$$\therefore \text{since } AD = f \binom{1}{1}^p, \quad AD' = f \binom{1}{-1}^p;$$

$$\therefore a + b \binom{1}{-1}^m + c \binom{1}{-1}^n = f \binom{1}{-1}^p.$$

(115.) *If $a + b \binom{1}{1}^m + c \binom{1}{1}^n = f \binom{1}{1}^p$, and a , b , c , f be either positive or negative quantities;*

$$a + b \binom{1}{-1}^m + c \binom{1}{-1}^n = f \binom{1}{-1}^p.$$

For let one of the quantities b be negative, let $b = -g$;

$$\text{then } a - g \binom{1}{1}^m + c \binom{1}{1}^n = f \binom{1}{1}^p;$$

$$\text{Now } -g = g \left(\frac{1}{1} \right)^{\frac{1}{2}},$$

$$\therefore -g \left(\frac{1}{1} \right)^m = g \left(\frac{1}{1} \right)^{\frac{1}{2}} \cdot \left(\frac{1}{1} \right)^m = g \left(\frac{1}{1} \right)^{m+\frac{1}{2}},$$

$$\therefore a + g \left(\frac{1}{1} \right)^{m+\frac{1}{2}} + c \left(\frac{1}{1} \right)^n = f \left(\frac{1}{1} \right)^p,$$

$$\therefore a + g \left(\frac{1}{-1} \right)^{m+\frac{1}{2}} + c \left(\frac{1}{-1} \right)^n = f \left(\frac{1}{-1} \right)^p \text{ (by Art. 114.)}$$

$$\text{But } g \left(\frac{1}{-1} \right)^{m+\frac{1}{2}} = g \left(\frac{1}{-1} \right)^{\frac{1}{2}} \cdot \left(\frac{1}{-1} \right)^m = -g \left(\frac{1}{-1} \right)^m,$$

$$\therefore a - g \left(\frac{1}{-1} \right)^m + c \left(\frac{1}{-1} \right)^n = f \left(\frac{1}{-1} \right)^p,$$

$$\text{or } a + b \left(\frac{1}{-1} \right)^m + c \left(\frac{1}{-1} \right)^n = f \left(\frac{1}{-1} \right)^p;$$

In like manner it may be proved if any other of the quantities be negative.

(116.) *If* $a + b \left(\frac{1}{1} \right)^m + c \left(\frac{1}{1} \right)^n = f \left(\frac{1}{1} \right)^p$, *and* a, b, c, f , *be either positive or negative quantities,*

$$a + b \left(\frac{1}{1} \right)^{-m} + c \left(\frac{1}{1} \right)^{-n} = f \left(\frac{1}{1} \right)^{-p}.$$

For $a + b \left(\frac{1}{-1} \right)^m + c \left(\frac{1}{-1} \right)^n = f \left(\frac{1}{-1} \right)^p$ (by Art. 115.)

$$\text{And } \left(\frac{1}{-1} \right)^m = \left(\frac{1}{1} \right)^{-m},$$

$$\therefore a + b \left(\frac{1}{1} \right)^{-m} + c \left(\frac{1}{1} \right)^{-n} = f \left(\frac{1}{1} \right)^{-p}.$$

(117.) *If* $a + b\sqrt{-1} + c\left(\frac{1}{1}\right)^n = f\left(\frac{1}{1}\right)^p$, *and* a , b , c , f *be either positive or negative quantities,*

$$a - b\sqrt{-1} + c\left(\frac{1}{1}\right)^{-n} = f\left(\frac{1}{1}\right)^{-p}.$$

For $+\sqrt{-1} = \left(\frac{1}{1}\right)^{\frac{1}{2}}$ (by Art. 107.)

$$\therefore a + b\left(\frac{1}{1}\right)^{\frac{1}{2}} + c\left(\frac{1}{1}\right)^n = f\left(\frac{1}{1}\right)^p,$$

$$\therefore a + b\left(\frac{1}{1}\right)^{-\frac{1}{2}} + c\left(\frac{1}{1}\right)^{-n} = f\left(\frac{1}{1}\right)^{-p};$$

But $\left(\frac{1}{1}\right)^{-\frac{1}{2}} = -\sqrt{-1}$ (by Art. 108.)

$$\therefore a - b\sqrt{-1} + c\left(\frac{1}{1}\right)^{-n} = f\left(\frac{1}{1}\right)^{-p}.$$

(118.) *If* $a + b\sqrt{-1} = e + f\sqrt{-1}$, *and* a , b , e , f , *be either positive or negative quantities; a = e and b = f.*

For since $a + b\sqrt{-1} = e + f\sqrt{-1}$,

$$a - b\sqrt{-1} = e - f\sqrt{-1} \text{ (by Art. 117.)}$$

\therefore by addition and subtraction,

$$2a = 2e,$$

$$\text{and } 2b\sqrt{-1} = 2f\sqrt{-1};$$

$$\therefore a = e,$$

$$\text{and } b = f.$$

(119.) *If $a = \pm b \pm c\sqrt{-1}$, where b and c are positive quantities, a will be in length $= \sqrt{b^2 + c^2}$.*

For let $a = e \left(\frac{1}{1}\right)^n$, where e is a positive quantity,

$$\text{then } e \left(\frac{1}{1}\right)^n = \pm b \pm c\sqrt{-1},$$

$$\therefore e \left(\frac{1}{1}\right)^{-n} = \pm b \mp c\sqrt{-1} \text{ (by Art. 117.)}$$

\therefore by multiplication,

$$e^2 = b^2 + c^2,$$

$$e = \sqrt{b^2 + c^2},$$

$$\therefore a \text{ is in length } = \sqrt{b^2 + c^2}.$$

(120.) *Let $a = b \left(\frac{1}{1}\right)^n$, where b is a positive quantity, and n positive and less than 1; then if n be either not greater than $\frac{1}{4}$, or not less than $\frac{3}{4}$, $a + 1$ is greater in length than 1.*

For since $a = b \left(\frac{1}{1}\right)^n$, and n is either not greater than $\frac{1}{4}$ or not less than $\frac{3}{4}$, a is of the form $e \pm f\sqrt{-1}$ (by Art. 111), where $e = 0$, or some positive quantity, and f is a positive quantity;

$$\therefore a + 1 = 1 + e \pm f\sqrt{-1},$$

$\therefore a + 1$ is in length equal to $\sqrt{(1+e)^2 + f^2}$;

$\therefore a + 1$ is in length greater than 1.

(121.) Let $a = b \left(\frac{1}{1}\right)^n$, where b is a positive quantity, and n not less than $\frac{1}{4}$ and not greater than $\frac{3}{4}$; then $a - 1$ is in length greater than 1.

For a is of the form $-e \pm f\sqrt{-1}$ (by Art. 111), where e and f either = 0, or are positive quantities;

$$\therefore a - 1 = -1 - e \pm f\sqrt{-1},$$

$\therefore a - 1$ is in length equal to $\sqrt{(1+e)^2 + f^2}$;

$\therefore a - 1$ is in length greater than 1.

(122.) Let $a = b \left(\frac{1}{1}\right)^n$, where b is a positive quantity, and n positive and less than 1; then if n be either less than $\frac{1}{4}$, or greater than $\frac{3}{4}$, $\frac{a-1}{a+1}$ is in length less than 1; but if n be greater than $\frac{1}{4}$, and less than $\frac{3}{4}$, $\frac{a-1}{a+1}$ is in length greater than 1.

First let n be either less than $\frac{1}{4}$, or greater than $\frac{3}{4}$;

Then a is of the form $e \pm f\sqrt{-1}$, where e and f are positive quantities;

$\therefore a-1$ is in length equal to $\sqrt{(e-1)^2 + f^2}$,

and $a+1$ $\sqrt{(e+1)^2 + f^2}$,

$\therefore \frac{a-1}{a+1}$ $\frac{\sqrt{(e-1)^2 + f^2}}{\sqrt{(e+1)^2 + f^2}}$;

$\therefore \frac{a-1}{a+1}$ is in length less than 1.

Next let n be greater than $\frac{1}{4}$, and less than $\frac{3}{4}$,

Then a is of the form $-e \pm f\sqrt{-1}$,

$\therefore \frac{a-1}{a+1}$ is in length equal to $\frac{\sqrt{(e+1)^2 + f^2}}{\sqrt{(e-1)^2 + f^2}}$;

$\therefore \frac{a-1}{a+1}$ is in length greater than 1.

(123.) $\left(\frac{1}{1}\right)^n + \left(\frac{1}{1}\right)^{-n}$ is equal to a positive quantity, when n is positive and less than $\frac{1}{4}$.

For if n be positive and less than $\frac{1}{4}$, $\left(\frac{1}{1}\right)^n$ is of the form $a + b\sqrt{-1}$, where a and b are positive quantities,

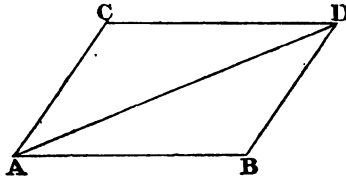
$\therefore \left(\frac{1}{1}\right)^{-n} = a - b\sqrt{-1}$ (by Art. 117),

$\therefore \left(\frac{1}{1}\right)^n + \left(\frac{1}{1}\right)^{-n} = 2a$.

(124.) Let $a = b \left(\frac{1}{1}\right)^m$ and $a + 1 = c \left(\frac{1}{1}\right)^p$, where b and c are positive quantities, and m and p positive and less than 1;

Then if m be less than $\frac{1}{2}$, p will be less than m ,
greater.....,greater.....

First let m be less than $\frac{1}{2}$,



And let $AB = \text{unity}$, $AC = a$, and let the parallelogram $ABDC$ be completed, and the diagonal AD be drawn;

Then $AD = a + 1 = c \left(\frac{1}{1}\right)^p$;

\therefore angle $BAD = p \cdot 360^\circ$;

and angle $BAC = m \cdot 360^\circ$;

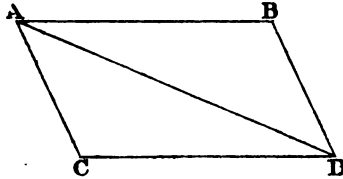
$\therefore p$ is less than m .

Next let m be greater than $\frac{1}{2}$,

The same construction being made, AC falls on the other side of AB ;

$$\therefore \text{angle } BAD = \overline{1-p} \cdot 360^\circ,$$

$$\dots\dots\dots BAC = \overline{1-m} \cdot 360^\circ;$$



$$\therefore 1-m \text{ is greater than } 1-p;$$

$$\therefore p \text{ is greater than } m.$$



CHAP. III.

BINOMIAL THEOREM, EXPANSION OF a^x IN A SERIES
ARRANGED ACCORDING TO THE POWERS OF x ,
DIFFERENTIATION OF a^x .

(125.) *If $a = 1 + b$, it is proved in the binomial theorem that*

$$a^m = 1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{1 \cdot 2 \cdot 3} b^3 + \&c.$$

which is a converging series when b is in length less than unity;

Now if m be not a whole number, a^m has many values;

To investigate which of the values of a^m is represented by the series

$$1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{1 \cdot 2 \cdot 3} b^3 + \&c.$$

First let a be a positive quantity,

then $(a)_0^m$ is also a positive quantity;

$\therefore (a)_0^m$ is that value of a^m which is represented by the series

$$1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.$$

Next let $a = c (1)_1^n$, where c is a positive quantity and n positive and less than $\frac{1}{2}$;

When $b = 0$, the series

$$1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c. = 1;$$

\therefore when $b = 0$, that value of a^m which is represented by $1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.$ must $= 1$;

But 1 is inclined to unity at an angle $= 0$;

\therefore when $b = 0$, that value of a^m which is represented by $1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.$ must be inclined to unity at an angle $= 0$;

Now (by Art. 86) $(a)_0^m = (c)_0^m \cdot (1)_1^{mn}$;

$\therefore (a)_0^m$ is inclined to unity at an angle $= mn \cdot 360^\circ$;

And a is of the form $e + f\sqrt{-1}$ (by Art. 111), where e and f are positive quantities,

$$\therefore b = e - 1 + f\sqrt{-1};$$

\therefore as b decreases, e approaches to 1 and f to 0,

$$\therefore \frac{f}{e} \text{ decreases,}$$

\therefore (by Art. 112.) n decreases,

And when $b = 0$, $f = 0$, and $e = 1$,

$$\therefore \frac{f}{e} = 0, \therefore n = 0,$$

$$\therefore mn 360^\circ = 0,$$

\therefore when $b = 0$, $(a)_0^m$ is inclined to unity at an angle $= 0$;

$$\therefore (a)_0^m \text{ is that value of } a^m \text{ which} = 1 + mb + \frac{m \cdot m - 1}{1 \cdot 2} b^2 + \&c.$$

Next let n be not less than $\frac{1}{4}$ and not greater than $\frac{3}{4}$;

In this case, since $b = a - 1$, b is always in length greater than 1 (by Art. 121.)

\therefore the series $1 + mb + \frac{m \cdot m - 1}{1 \cdot 2} b^2 + \&c.$ is not a converging series, and therefore it does not represent any value of a^m .

Next let n be greater than $\frac{3}{2}$ and less than 1 ;

Then a is of the form $e - f\sqrt{-1}$,

And $b = e - 1 - f\sqrt{-1}$;

\therefore as b decreases, $\frac{f}{e}$ decreases,

\therefore (by Art. 112.) n increases,

And when $b = 0$, $\frac{f}{e} = 0$, and $n = 1$;

$$\text{Now } \left(\frac{a}{-1}\right)^m = \left(\frac{a}{0}\right)^m \left(\frac{1}{-1}\right)^m = \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{-m}$$

$$= \left(\frac{c}{0}\right)^m \left(\frac{1}{1}\right)^{mn} \cdot \left(\frac{1}{1}\right)^{-m}$$

$$= \left(\frac{c}{0}\right)^m \left(\frac{1}{1}\right)^{m \cdot (n-1)}$$

$\therefore \left(\frac{a}{-1}\right)^m$ is inclined to unity at an angle

$$= m \cdot \overline{n-1} \cdot 360^\circ ;$$

But when $b = 0$, $n = 1$; $\therefore m \cdot \overline{n-1} \cdot 360^\circ = 0$,

\therefore when $b = 0$, $\left(\frac{a}{-1}\right)^m$ is inclined to unity at an angle $= 0$;

\therefore in this case $\left(\frac{a}{-1}\right)^m$ is that value of a^m which

$$= 1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.$$

(126.) Let $a = c \left(\frac{1}{1}\right)^n$, where c is a positive quantity, and n positive and less than 1, and let $a - 1 = b$, where b is in length less than unity;

Then, if n be less than $\frac{1}{4}$,

$$\left(\frac{a}{p}\right)^m = \left(\frac{1}{1}\right)^{pm} \cdot \left\{1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.\right\};$$

If n be greater than $\frac{3}{4}$,

$$\left(\frac{a}{p}\right)^m = \left(\frac{1}{1}\right)^{\overline{p+1} \cdot m} \cdot \left\{1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.\right\}.$$

First let n be less than $\frac{1}{4}$;

Then (by Art. 125),

$$\left(\frac{a}{0}\right)^m = 1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.$$

$$\text{But } \left(\frac{a}{p}\right)^m = \left(\frac{1}{p}\right)^m \cdot \left(\frac{a}{0}\right)^m = \left(\frac{1}{1}\right)^{mp} \cdot \left(\frac{a}{0}\right)^m;$$

$$\therefore \left(\frac{a}{p}\right)^m = \left(\frac{1}{1}\right)^{pm} \cdot \left\{1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.\right\}.$$

Next let n be greater than $\frac{3}{4}$,

$$(\text{By Art. 125}), \left(\frac{a}{-1}\right)^m = 1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.$$

$$\therefore \left(\frac{a}{0}\right)^m \cdot \left(\frac{1}{1}\right)^{-m} = 1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c.$$

$$\therefore (a)_m^m = (1)_1^m \left\{ 1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c. \right\},$$

$$\therefore (a)_p^m = (1)_{p+1}^{p+1 \cdot m} \left\{ 1 + mb + \frac{m \cdot \overline{m-1}}{1 \cdot 2} b^2 + \&c. \right\}.$$

(127.) *Let $1 + Ax + Bx^2 + Cx^3$ be a series such that*

$$(1 + Ax + Bx^2 + \&c.) \cdot (1 + Ay + By^2 + \&c.) \\ = 1 + A \cdot (x + y) + B(x + y)^2 + \&c.$$

whatever be the values of x and y ;

To find the law of the series.

By multiplication,

$$(1 + Ax + Bx^2 + \&c.) \cdot (1 + Ay + By^2 + \&c.) = \\ = \begin{cases} 1 + Ax + Bx^2 + Cx^3 + \&c. \\ + Ay + A^2xy + ABx^2y + \&c. \\ + By^2 + ABxy^2 + \&c. \\ + Cy^3 + \&c. \end{cases}$$

And by expansion,

$$1 + A(x + y) + B(x + y)^2 + \&c. = \\ = \left\{ 1 + \begin{matrix} Ax \\ + Ay \end{matrix} + \begin{matrix} Bx^2 \\ + 2Bxy \\ + By^2 \end{matrix} + \begin{matrix} Cx^3 \\ + 3Cx^2y \\ + 3Cxy^2 \\ + Cy^3 \end{matrix} \right\} + \&c.$$

\therefore equating the coefficients of the combinations of the like powers of x and y ,

$$2B = A^2; \therefore B = \frac{A^2}{2},$$

$$3C = AB; \therefore C = \frac{AB}{3} = \frac{A^3}{2.3};$$

$\therefore 1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{1.2.3} + \&c.$ is a series such that

$$\begin{aligned} & \left(1 + Ax + \frac{A^2 x^2}{1.2} + \&c.\right) \cdot \left(1 + Ay + \frac{A^2 y^2}{1.2} + \&c.\right) \\ &= 1 + A(x + y) + \frac{A^2 (x + y)^2}{1.2} + \&c. \end{aligned}$$

whatever be the values of x , y , or A .

(128.) Let $a = c \left(\frac{1}{1}\right)^n$, where c is a positive quantity, and n positive and less than 1, and let $a - 1$ be in length less than unity, and $A = (a - 1) - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \&c.$

Then if n be less than $\frac{1}{4}$,

$$\left(\frac{a}{1}\right)^x = 1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{1.2.3} + \&c.$$

If n be greater than $\frac{3}{4}$,

$$\left(\frac{a}{1}\right)^x = 1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{1.2.3} + \&c.$$

For it is proved in the Appendix to Woodhouse's Trigonometry, that

$$a^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3} + \&c.$$

And the proof given by Woodhouse depends upon the binomial theorem ;

\therefore (by Art. 125.) If n be less than $\frac{1}{4}$,

$$\binom{a}{0}^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3} ;$$

If n be greater than $\frac{3}{4}$,

$$\binom{a}{-1}^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3}.$$

(129.) To express $\binom{1}{1}^x$ in a series of the form $a + bx + cx^2 + \&c.$

$$\text{Let } \binom{1}{1}^x = a + bx + cx^2 + \&c.$$

$$\text{Let } x = 0, \text{ then } \binom{1}{1}^0 = a,$$

$$\therefore 1 = a ;$$

$$\therefore \binom{1}{1}^x = 1 + bx + cx^2 + \&c.$$

$$\text{Now } \binom{1}{1}^x \cdot \binom{1}{1}^y = \binom{1}{1}^{x+y} ;$$

$\therefore 1 + bx + cx^2 + \&c.$ is a series such that

$$(1 + bx + cx^2 + \&c.) \cdot (1 + by + cy^2 + \&c.) \\ = 1 + b \cdot (x + y) + c (x + y)^2 + \&c.$$

$$\therefore (\text{by Art. 127.}) \quad c = \frac{b^2}{2}, \quad \&c.$$

$$\therefore \binom{1}{1}^x = 1 + bx + \frac{b^2 x^2}{1 \cdot 2} + \&c.$$

To find the value of b ,

Since x may be any value, let x be positive and less than $\frac{1}{4}$;

$$\text{And let } \binom{1}{1}^x = \frac{1 + a}{1 - a},$$

$$\text{Also let } \binom{1}{1}^x = k, \quad 1 + a = m, \quad 1 - a = n,$$

$$\text{then } \frac{m}{n} = k;$$

$$\text{and } a = \frac{\binom{1}{1}^x - 1}{\binom{1}{1}^x + 1}, \quad \text{where since } x \text{ is positive and less}$$

than $\frac{1}{4}$, a is in length less than unity (by Art. 122),

$$\text{and } m = 1 + \frac{\binom{1}{1}^x - 1}{\binom{1}{1}^x + 1} = \frac{2 \binom{1}{1}^x}{\binom{1}{1}^x + 1} = \frac{2 \binom{1}{1}^{\frac{x}{2}}}{\binom{1}{1}^{\frac{x}{2}} + \binom{1}{1}^{-\frac{x}{2}}};$$

$$\text{Let } \frac{2}{\binom{1}{1}^{\frac{x}{2}} + \binom{1}{1}^{-\frac{x}{2}}} = g,$$

then g is a positive quantity (by Art. 123.)

$$\therefore m = g \binom{1}{1}^{\frac{x}{2}},$$

$$\therefore n = \frac{g \binom{1}{1}^{\frac{x}{2}}}{\binom{1}{1}^x} = g \binom{1}{1}^{-\frac{x}{2}} = g \binom{1}{1}^{1-\frac{x}{2}};$$

$\therefore m$ is inclined to unity at an angle $= \frac{x}{2} 360^\circ$,

$$n \dots \dots \dots = \left(1 - \frac{x}{2}\right) \cdot 360^\circ;$$

And $1 - \frac{x}{2}$ is greater than $\frac{x}{2}$, since x is less than $\frac{1}{4}$,

$$\therefore (\text{by Art. 93.}) \frac{\binom{m}{0}^\nu}{\binom{n}{0}^\nu} = \binom{k}{-1}^\nu = \binom{k}{0}^\nu \cdot \binom{1}{1}^{-\nu};$$

$$\therefore \binom{k}{0}^\nu = \frac{\binom{m}{0}^\nu}{\binom{n}{0}^\nu \cdot \binom{1}{1}^{-\nu}} = \frac{\binom{m}{0}^\nu}{\binom{n}{-1}^\nu};$$

$$\text{But } k = \binom{1}{1}^x,$$

\therefore since x is positive and less than 1,

$$\binom{k}{0}^\nu = \binom{1}{1}^{x\nu} \text{ (by Art. 86.)}$$

$$\therefore \binom{1}{1}^{xy} = \frac{\binom{m}{0}^y}{\binom{n}{-1}^y} = \binom{m}{0}^y \cdot \binom{n}{-1}^{-y};$$

Since $m = g \left(1\right)^{\frac{x}{2}}$, and $\frac{x}{2}$ is positive and less than $\frac{1}{4}$,

$$\binom{m}{0}^y = 1 + My + \frac{M^2 y^2}{1 \cdot 2} + \&c. \text{ (by Art. 128),}$$

where $M = (m-1) - \frac{1}{2}(m-1)^2 + \frac{1}{3}(m-1)^3 - \&c.$

And since $n = g \left(1\right)^{1-\frac{x}{2}}$, and $1 - \frac{x}{2}$ is greater than $\frac{3}{4}$ and less than 1,

$$\binom{n}{-1}^{-y} = 1 - Ny + \frac{N^2 y^2}{1 \cdot 2} - \&c. \text{ (by Art. 128),}$$

where $N = (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.$

$$\therefore \binom{1}{1}^{xy} = \left(1 + My + \frac{M^2 y^2}{1 \cdot 2} + \&c.\right) \cdot \left(1 - Ny + \frac{N^2 y^2}{1 \cdot 2} - \&c.\right)$$

$$= 1 + (M-N)y + \frac{(M-N)^2 y^2}{1 \cdot 2} + \&c. \text{ (by Art. 127.)}$$

$$\therefore 1 + bxy + \frac{b^2 x^2 y^2}{1 \cdot 2} + \&c.$$

$$= 1 + (M-N)y + \frac{(M-N)^2 y^2}{1 \cdot 2} + \&c.$$

$$\therefore bx = M-N,$$

$$\therefore b = \frac{1}{x} \cdot (M-N);$$

Now $m = 1 + a$, $n = 1 - a$,

$$\therefore m - 1 = a, \quad n - 1 = -a;$$

$$\therefore M = a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{4}a^4 + \frac{1}{5}a^5 - \&c.$$

$$N = -a - \frac{1}{2}a^2 - \frac{1}{3}a^3 - \frac{1}{4}a^4 - \frac{1}{5}a^5 - \&c.$$

$$\therefore M - N = 2 \left\{ a + \frac{1}{3}a^3 + \frac{1}{5}a^5 + \&c. \right\}$$

$$\therefore b = \frac{2}{x} \cdot \left\{ a + \frac{1}{3}a^3 + \frac{1}{5}a^5 + \&c. \right\}$$

$$\text{Let } x = \frac{1}{6},$$

$$\text{Then } \left(1\right)^{\frac{1}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \sqrt{-1} \text{ (by Art. 113.)}$$

$$\therefore a = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1} - 1}{\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1} + 1} = \frac{-1 + \sqrt{3} \cdot \sqrt{-1}}{3 + \sqrt{3} \cdot \sqrt{-1}}$$

$$= \frac{(\sqrt{3} + \sqrt{-1})\sqrt{-1}}{3 + \sqrt{3} \cdot \sqrt{-1}} = \frac{\sqrt{-1}}{\sqrt{3}};$$

$$\therefore b = 2 \cdot 6 \cdot \left\{ \frac{\sqrt{-1}}{\sqrt{3}} + \frac{1}{3} \cdot \left(\frac{\sqrt{-1}}{\sqrt{3}} \right)^3 + \frac{1}{5} \cdot \left(\frac{\sqrt{-1}}{\sqrt{3}} \right)^5 + \&c. \right\}$$

$$= 12 \sqrt{-1} \left\{ \frac{1}{\sqrt{3}} - \frac{1}{3} \cdot \left(\frac{1}{\sqrt{3}} \right)^3 + \frac{1}{5} \left(\frac{1}{\sqrt{3}} \right)^5 - \&c. \right\},$$

$$\text{Let } 12 \left\{ \frac{1}{\sqrt{3}} - \frac{1}{3} \left(\frac{1}{\sqrt{3}} \right)^3 + \frac{1}{5} \left(\frac{1}{\sqrt{3}} \right)^5 - \&c. \right\} = c,$$

then $b = c\sqrt{-1}$, where c is a positive quantity ;

$$\therefore \left(\frac{1}{1} \right)^x = 1 + cx\sqrt{-1} - \frac{c^2 x^2}{1.2} - \frac{c^3 x^3}{1.2.3} \sqrt{-1} + \&c.$$

$$(130.) \text{ COR. 1. } \left(\frac{1}{p} \right)^x$$

$$= 1 + pcx\sqrt{-1} - \frac{p^2 c^2 x^2}{1.2} - \frac{p^3 c^3 x^3}{1.2.3} \sqrt{-1} + \&c.$$

$$\text{For } \left(\frac{1}{p} \right)^x = \left(\frac{1}{1} \right)^{px}$$

$$= 1 + pcx\sqrt{-1} - \frac{p^2 c^2 x^2}{1.2} - \frac{p^3 c^3 x^3}{1.2.3} \sqrt{-1} + \&c.$$

(131.) COR. 2. If a be a positive quantity

$$\begin{aligned} \left(\frac{a}{p} \right)^x &= 1 + (A + pc\sqrt{-1})x + \frac{(A + pc\sqrt{-1})^2 x^2}{1.2} \\ &\quad + \frac{(A + pc\sqrt{-1})^3 x^3}{1.2.3} + \&c. \end{aligned}$$

$$\text{where } A = 2 \left\{ \frac{a-1}{a+1} + \frac{1}{3} \left(\frac{a-1}{a+1} \right)^3 + \frac{1}{5} \left(\frac{a-1}{a+1} \right)^5 + \&c. \right\}.$$

For by the Appendix to Woodhouse's Trigonometry,

$$\binom{a}{0}^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$\text{where } A = 2 \left\{ \frac{a-1}{a+1} + \frac{1}{3} \left(\frac{a-1}{a+1} \right)^3 + \frac{1}{5} \left(\frac{a-1}{a+1} \right)^5 + \&c. \right\}$$

$$\begin{aligned} \therefore \binom{a}{p}^x &= \left(1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \&c. \right) \cdot \binom{1}{p}^x \\ &= \left(1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \&c. \right) \cdot \left(1 + pcx\sqrt{-1} + \frac{(pcx\sqrt{-1})^2}{1 \cdot 2} + \&c. \right) \\ &= 1 + (A + pc\sqrt{-1})x + \frac{(A + pc\sqrt{-1})^2 x^2}{1 \cdot 2} + \&c. \text{ (by Art. 127.)} \end{aligned}$$

(132.) Let $a = b \binom{1}{1}^p$, where b is a positive quantity, and p positive and less than 1;

Then if p be less than $\frac{1}{4}$,

$$\binom{a}{0}^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \&c.$$

Then if p be greater than $\frac{3}{4}$,

$$\binom{a}{-1}^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \&c.$$

$$\text{where } A = 2 \left\{ \frac{a-1}{a+1} + \frac{1}{3} \left(\frac{a-1}{a+1} \right)^3 + \frac{1}{5} \left(\frac{a-1}{a+1} \right)^5 + \&c. \right\}$$

For let $a = \frac{1+e}{1-e}$, and let $1+e=m$, $1-e=n$,

$$\text{then } e = \frac{a-1}{a+1}, \quad m = \frac{2a}{a+1}, \quad n = \frac{2}{a+1};$$

Let $a+1 = f\left(\frac{1}{1}\right)^q$, where f is a positive quantity,
and q positive and less than 1,

$$\text{then } m = \frac{2b\left(\frac{1}{1}\right)^p}{f\left(\frac{1}{1}\right)^q} = \frac{2b}{f} \cdot \left(\frac{1}{1}\right)^{p-q},$$

$$n = \frac{2}{f\left(\frac{1}{1}\right)^q} = \frac{2}{f} \cdot \left(\frac{1}{1}\right)^{-q} = \frac{2}{f} \cdot \left(\frac{1}{1}\right)^{1-q};$$

First let p be less than $\frac{1}{4}$,

Then q is less than p (by Art. 124),

$\therefore p - q$ is positive and less than $\frac{1}{4}$, and $1 - q$ is
greater than $\frac{3}{4}$ and less than 1;

Also m is inclined to unity at an angle $= \overline{p-q} \cdot 360^\circ$,

$$n \dots \dots \dots = \overline{1-q} \cdot 360^\circ,$$

And $p - q$ is less than $1 - q$,

$$\therefore \binom{a}{-1}^x = \frac{\binom{m}{0}^x}{\binom{n}{0}^x} \quad (\text{by Art. 93.})$$

$$\text{or } \binom{a}{0}^x \cdot \binom{1}{-1}^x = \frac{\binom{m}{0}^x}{\binom{n}{0}^x},$$

$$\begin{aligned}\therefore \binom{a}{0}^x &= \frac{\binom{m}{0}^x}{\binom{m}{0}^x \binom{1}{-1}^x} = \frac{\binom{m}{0}^x}{\binom{n}{-1}^x} = \binom{m}{0}^x \cdot \binom{n}{-1}^{-x} \\ &= \left(1 + Mx + \frac{M^2 x^2}{1 \cdot 2} + \&c.\right).\end{aligned}$$

$$\left(1 - Nx + \frac{N^2 x^2}{1 \cdot 2} - \&c.\right) \text{ (by Art. 128.)}$$

$$\text{where } M = (m-1) - \frac{1}{2}(m-1)^2 + \frac{1}{3}(m-1)^3 - \&c.$$

$$N = (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.$$

$$\therefore \binom{a}{0}^x = 1 + (M-N)x + \frac{(M-N)^2 x^2}{1 \cdot 2} + \&c.$$

$$\text{But } m = 1 + e, \quad n = 1 - e;$$

$$\therefore M = e - \frac{1}{2}e^2 + \frac{1}{3}e^3 - \frac{1}{4}e^4 + \frac{1}{5}e^5 - \&c.$$

$$N = -e - \frac{1}{2}e^2 - \frac{1}{3}e^3 - \frac{1}{4}e^4 - \frac{1}{5}e^5 - \&c.$$

$$\therefore M - N = 2 \left\{ e + \frac{1}{3}e^3 + \frac{1}{5}e^5 + \&c. \right\}$$

$$= 2 \left\{ \frac{a-1}{a+1} + \frac{1}{3} \left(\frac{a-1}{a+1} \right)^3 + \frac{1}{5} \left(\frac{a-1}{a+1} \right)^5 + \&c. \right\};$$

$$\therefore \binom{a}{0}^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \&c.$$

$$\text{where } A = 2 \left\{ \frac{a-1}{a+1} + \frac{1}{3} \left(\frac{a-1}{a+1} \right)^3 + \frac{1}{5} \left(\frac{a-1}{a+1} \right)^5 + \&c. \right\}$$

which is a converging series (by Art. 122).

Next let p be greater than $\frac{3}{4}$,

Then q is greater than p (by Art. 124.)

$\therefore p - q$ is negative and between $-\frac{1}{4}$ and 0;

$$\text{But } m = \frac{2b}{f} \cdot \left(\frac{1}{1}\right)^{p-q} = \frac{2b}{f} \cdot \left(\frac{1}{1}\right)^{1+p-q},$$

where $1 + p - q$ is between $\frac{3}{4}$ and 1;

Also $1 - q$ is between 0 and $\frac{1}{4}$;

And m is inclined to unity at an angle $= \overline{1 + p - q} \cdot 360^\circ$,

$$n \dots \dots \dots = \overline{1 - q} \cdot 360^\circ,$$

And $1 + p - q$ is greater than $1 - q$,

$$\therefore \binom{a}{0}^z = \frac{\binom{m}{0}^z}{\binom{n}{0}^z} \text{ (by Art. 93);}$$

$$\therefore \binom{a}{-1}^z = \frac{\binom{m}{-1}^z}{\binom{n}{0}^z} = \binom{m}{-1}^z \cdot \binom{n}{0}^{-z}$$

$$= \left(1 + Mx + \frac{M^2 x^2}{1 \cdot 2} + \&c.\right).$$

$$\left(1 - Nx + \frac{N^2 x^2}{1 \cdot 2} - \&c.\right) \text{ (by Art. 128.)}$$

$$= 1 + (M - N)x + \frac{(M - N)^2 x^2}{1 \cdot 2} + \&c.$$

$$= 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \&c.$$

(133.) Let $a = b \binom{1}{1}^p$, where b is a positive quantity, and p positive and less than 1; then if p be greater than $\frac{1}{4}$ and less than $\frac{3}{4}$,

$$\binom{a}{0}^x = 1 + \left(A' + \frac{c}{2} \sqrt{-1}\right)x + \frac{\left(A' + \frac{c}{2} \sqrt{-1}\right)^2 x^2}{1 \cdot 2} + \&c.$$

$$\text{where } A' = 2 \left\{ \frac{a+1}{a-1} + \frac{1}{3} \left(\frac{a+1}{a-1}\right)^3 + \frac{1}{5} \left(\frac{a+1}{a-1}\right)^5 + \&c. \right\}$$

and c is of the same value as in Art. 129.

For first let p be less than $\frac{1}{2}$.

Let $f = -a$,

$$\text{then } f = -1 \cdot a = \binom{1}{1}^{\frac{1}{2}} \cdot a = \binom{1}{1}^{\frac{1}{2}} b \binom{1}{1}^p = b \binom{1}{1}^{p+\frac{1}{2}},$$

where since p is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$,

$$p + \frac{1}{2} \dots \dots \dots \frac{3}{4} \dots \dots \dots 1;$$

$$\therefore \binom{f}{-1}^x = 1 + A'x + \frac{A'^2 x^2}{1 \cdot 2} + \&c.$$

$$\text{where } A' = 2 \left\{ \frac{f-1}{f+1} + \frac{1}{3} \left(\frac{f-1}{f+1}\right)^3 + \frac{1}{5} \left(\frac{f-1}{f+1}\right)^5 + \&c. \right\}$$

(by Art. 132.)

$$\text{But } \frac{f-1}{f+1} = \frac{-a-1}{-a+1} = \frac{a+1}{a-1}$$

a proper fraction (by Art. 122.)

$$\therefore A' = 2 \left\{ \frac{a+1}{a-1} + \frac{1}{3} \left(\frac{a+1}{a-1} \right)^3 + \frac{1}{5} \left(\frac{a+1}{a-1} \right)^5 + \&c. \right\}$$

But $f = -1 \cdot a$,

$$\therefore (f)_0^x = (-1)_0^x \cdot (a)_0^x \text{ (by Art. 91.)}$$

$$= \left(1_1\right)^{\frac{x}{2}} \cdot (a)_0^x \text{ (by Art. 86.)}$$

$$\therefore (f)_{-1}^x = \left(1_1\right)^{\frac{x}{2}} \cdot (a)_0^x \cdot (1)_1^{-x} = (a)_0^x \cdot \left(1_1\right)^{-\frac{x}{2}},$$

$$\therefore (a)_0^x = (f)_{-1}^x \cdot \left(1_1\right)^{\frac{x}{2}}$$

$$= \left(1 + A'x + \frac{A'^2 x^2}{1 \cdot 2} + \&c. \right).$$

$$\left\{ 1 + \frac{cx}{2} \sqrt{-1} + \frac{\left(c \frac{x}{2} \sqrt{-1}\right)^2}{1 \cdot 2} + \&c. \right\}$$

$$= 1 + \left(A' + \frac{c}{2} \sqrt{-1}\right) x + \frac{\left(A' + \frac{c}{2} \sqrt{-1}\right)^2 x^2}{1 \cdot 2} + \&c.$$

Next let p be greater than $\frac{1}{2}$;

As in the first case, let $f = -a$,

$$\text{then } f = b \left(1_1\right)^{p+\frac{1}{2}} = b \left(1_1\right)^{p+\frac{1}{2}-1} = b \left(1_1\right)^{p-\frac{1}{2}},$$

where since p is greater than $\frac{1}{2}$ and less than $\frac{3}{4}$,

$$p - \frac{1}{2} \dots\dots\dots 0 \dots\dots\dots \frac{1}{4};$$

$$\therefore (f)_s = 1 + A'x + \frac{A'^2 x^2}{1.2} + \&c.$$

$$\text{But } f = -1.a;$$

$$\therefore \text{ in this case } (f)_s = (-1)_s \cdot (a)_s \text{ (by Art. 91.)}$$

$$= (1)_s^{\frac{2}{3}} \cdot (a)_s;$$

$$\therefore (a)_s = (f)_s \cdot (1)_s^{-\frac{2}{3}} = (f)_s \cdot (1)_s^{\frac{2}{3}}$$

$$= \left(1 + A'x + \frac{A'^2 x^2}{1.2} + \&c.\right) \cdot (1)_s^{\frac{2}{3}}$$

$$= 1 + \left(A' + \frac{c}{2} \sqrt{-1}\right)x + \frac{\left(A' + \frac{c}{2} \sqrt{-1}\right)^2 x^2}{1.2} + \&c.$$

(134.) COR. Let $a = b (1)_s^p$, where b is a positive quantity and p positive and less than unity, and let $A = 2 \left\{ \frac{a-1}{a+1} + \frac{1}{3} \left(\frac{a-1}{a+1} \right)^3 + \frac{1}{5} \left(\frac{a-1}{a+1} \right)^5 + \&c. \right\}$,

$$A' = 2 \left\{ \frac{a+1}{a-1} + \frac{1}{3} \left(\frac{a+1}{a-1} \right)^3 + \frac{1}{5} \left(\frac{a+1}{a-1} \right)^5 + \&c. \right\},$$

and let c be of the same value as in Art. 129.

Then if p be less than $\frac{1}{4}$,

$$(a)_s = 1 + (A + qc \sqrt{-1})x + \frac{(A + qc \sqrt{-1})^2 x^2}{1.2} + \&c.$$

If p be greater than $\frac{1}{4}$ and less than $\frac{3}{4}$,

$$(a)_q^s = 1 + (A' + \overline{q + \frac{1}{2}} \cdot c \sqrt{-1})x + \frac{(A' + \overline{q + \frac{1}{2}} \cdot c \sqrt{-1})^2 x^2}{1 \cdot 2} + \&c.$$

If p be greater than $\frac{3}{4}$,

$$(a)_q^s = 1 + (A + \overline{q + 1} \cdot c \sqrt{-1})x + \frac{(A + \overline{q + 1} \cdot c \sqrt{-1})^2 x^2}{1 \cdot 2} + \&c.$$

For first let p be less than $\frac{1}{4}$,

$$\text{Then } (a)_q^s = (a)_0^s \cdot (1)_q^s = \left(1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \&c.\right).$$

$$\left(1 + qc x \sqrt{-1} + \frac{(qc x \sqrt{-1})^2}{1 \cdot 2} + \&c.\right)$$

$$= 1 + (A + qc \sqrt{-1})x + \frac{(A + qc \sqrt{-1})^2 x^2}{1 \cdot 2} + \&c.$$

Next let p be greater than $\frac{1}{4}$ and less than $\frac{3}{4}$,

$$\text{Then } (a)_q^s = (a)_0^s \cdot (1)_q^s =$$

$$\left\{1 + \left(A' + \frac{c}{2} \sqrt{-1}\right)x + \frac{\left(A' + \frac{c}{2} \sqrt{-1}\right)^2 x^2}{1 \cdot 2} + \&c.\right\}.$$

$$\left(1 + qc x \sqrt{-1} + \frac{(qc x \sqrt{-1})^2}{1 \cdot 2} + \&c.\right)$$

$$= 1 + (A' + \overline{q + \frac{1}{2}} \cdot c \sqrt{-1})x + \frac{(A' + \overline{q + \frac{1}{2}} \cdot c \sqrt{-1})^2 x^2}{1 \cdot 2} + \&c.$$

Next let p be greater than $\frac{3}{4}$,

$$\text{Then } (a)_q^s = (a)_0^s \cdot (1)_q^s$$

$$= (a)_{-1}^s \cdot (1)_1^s \cdot (1)_q^s = (a)_{-1}^s \cdot (1)^{\overline{s+1}.s}$$

$$= \left(1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \&c.\right).$$

$$\left(1 + \overline{q+1} \cdot cx \sqrt{-1} + \frac{(\overline{q+1} \cdot cx \sqrt{-1})^2}{1 \cdot 2} + \&c.\right)$$

$$= 1 + (A + \overline{q+1} \cdot c \sqrt{-1})x + \frac{(A + \overline{q+1} \cdot c \sqrt{-1})^2 x^2}{1 \cdot 2} + \&c.$$

(135.) Let $a = b \left(\frac{1}{1}\right)^p$, where b is a positive quantity, and p positive and less than 1,

$$\text{Then } (a)_q^s$$

$$= (B + \overline{p+q} \cdot c \sqrt{-1})x + \frac{(B + \overline{p+q} \cdot c \sqrt{-1})^2 x^2}{1 \cdot 2} + \&c.$$

$$\text{where } B = 2 \left\{ \frac{b-1}{b+1} + \frac{1}{3} \left(\frac{b-1}{b+1}\right)^3 + \frac{1}{5} \left(\frac{b-1}{b+1}\right)^5 + \&c. \right\},$$

and c is of the same value as in Art. 129.

$$\text{For } (b)_0^s = 1 + Bx + \frac{B^2 x^2}{1 \cdot 2} + \&c.$$

$$\text{and } \binom{a}{0}^r = \binom{b}{0}^r \cdot \binom{1}{1}^{rs};$$

$$\therefore \binom{a}{q}^r = \binom{b}{0}^r \cdot \binom{1}{1}^{rs} \cdot \binom{1}{1}^{qs} = \binom{b}{0}^r \cdot \binom{1}{1}^{\overline{r+q.s}}$$

$$= \left(1 + Bx + \frac{B^2 x^2}{1.2} + \&c. \right).$$

$$\left(1 + \overline{p+q}.cx\sqrt{-1} + \frac{(\overline{p+q}.cx\sqrt{-1})^2}{1.2} + \&c. \right)$$

$$= 1 + (B + \overline{p+q}.c\sqrt{-1})x + \frac{(B + \overline{p+q}.c\sqrt{-1})^2 x^2}{1.2} + \&c.$$

(136.) *To find the differential of $\binom{a}{q}^r$.*

Let $a = b \binom{1}{1}^p$, where b is a positive quantity,

and p positive and less than 1;

First let p be less than $\frac{1}{4}$,

$$\text{Let } \binom{a}{q}^r = u,$$

$$\binom{a}{q}^{r+h} = u',$$

$$u' - u = \binom{a}{q}^{r+h} - \binom{a}{q}^r = \binom{a}{q}^r \cdot \left\{ \binom{a}{q}^h - 1 \right\}$$

$$= \binom{a}{q}^r \left\{ 1 + (A + qc\sqrt{-1})h + \frac{(A + qc\sqrt{-1})^2 h^2}{1.2} + \&c. - 1 \right\},$$

$$\begin{aligned} \therefore \frac{u' - u}{h} \\ = u \left\{ A + qc \sqrt{-1} + \frac{(A + qc \sqrt{-1})^2 h}{1.2} + \&c. \right\}; \\ \therefore \frac{du}{dx} = u (A + qc \sqrt{-1}). \end{aligned}$$

Next let p be greater than $\frac{1}{4}$ and less than $\frac{3}{4}$,

It may be proved nearly as in the first case, that is this case,

$$\frac{du}{dx} = u (A + \overline{q + \frac{1}{2}} \cdot c \sqrt{-1}).$$

Next let p be greater than $\frac{3}{4}$,

In this case,

$$\frac{du}{dx} = u (A + \overline{q + 1} \cdot c \sqrt{-1}).$$

$$\text{Or if } B = 2 \left\{ \frac{b-1}{b+1} + \frac{1}{3} \left(\frac{b-1}{b+1} \right)^3 + \frac{1}{5} \left(\frac{b-1}{b+1} \right)^5 + \&c. \right\},$$

it may be proved in all cases that

$$\frac{du}{dx} = u (B + \overline{p + q} \cdot c \sqrt{-1}).$$

$$(137.) \text{ Cor. Hence if } u = \left(\frac{1}{1} \right)^x, \quad \frac{du}{dx} = u \epsilon \sqrt{-1}.$$

(138.) To integrate $\frac{du}{dx} = (a + b\sqrt{-1})u$.

Let $u = C \cdot (e)_0^{ms} \cdot (1)_1^{ns}$, where e is the base of the hyperbolic logarithms,

then $\frac{du}{dx}$

$$\begin{aligned} &= C \cdot (e)_0^{ms} \cdot m \cdot (1)_1^{ns} + C \cdot (e)_0^{ms} \cdot (1)_1^{ns} \cdot nc \sqrt{-1} \\ &= C \cdot (e)_0^{ms} \cdot (1)_1^{ns} \cdot (m + nc \sqrt{-1}) \\ &= u(m + nc \sqrt{-1}); \end{aligned}$$

$$\therefore m + nc \sqrt{-1} = a + b \sqrt{-1},$$

$$\therefore \text{(by Art. 118.) } m = a, \quad nc = b,$$

$$\therefore n = \frac{b}{c},$$

$$\therefore u = C \cdot (e)_0^{as} \cdot (1)_1^{\frac{b}{c}s};$$

where C is an arbitrary constant.

$$\text{If } b = 0, \quad \frac{du}{dx} = au,$$

$$\text{and } u = C \cdot (e)_0^{as},$$

$$\therefore ax = \text{hyp. log. } u - \text{hyp. log. } C.$$

$$\text{If } a = 0, \frac{du}{dx} = (b\sqrt{-1})u,$$

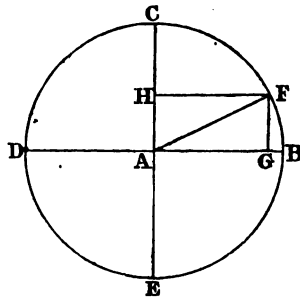
$$\text{and } u = C \cdot \left(\frac{1}{x}\right)^{\frac{b}{2}}.$$



CHAP. IV.

EXAMPLES ILLUSTRATIVE OF THE PRINCIPLES ESTABLISHED
IN THE PRECEDING CHAPTERS.

(139.) *To express the sines, cosines, &c. of
arcs in terms of the powers of unity.*



Let $AB = \text{unity}$, $BCDE$ be a circle described
with radius AB ;

Let BA be produced to D , and through A let
 EAC be drawn perpendicular to DB ;

Let BF be a given arc;

Join AF , and draw FG perpendicular to AB ,
and FH perpendicular to AC ;

Let $BF = \theta$, and $2\pi =$ whole circumference,

$$\text{Then } AF = \left(1_1\right)^{\frac{\theta}{2\pi}},$$

$$AG = \cos \theta, \quad AH = \sin \theta \sqrt{-1},$$

$$\therefore \left(1_1\right)^{\frac{\theta}{2\pi}} = \cos \theta + \sin \theta \sqrt{-1},$$

$$\therefore \left(1_1\right)^{-\frac{\theta}{2\pi}} = \cos \theta - \sin \theta \sqrt{-1} \quad (\text{by Art. 117.})$$

$$\therefore \text{ by addition } \left(1_1\right)^{\frac{\theta}{2\pi}} + \left(1_1\right)^{-\frac{\theta}{2\pi}} = 2 \cos \theta,$$

$$\text{By subtraction } \left(1_1\right)^{\frac{\theta}{2\pi}} - \left(1_1\right)^{-\frac{\theta}{2\pi}} = 2 \sin \theta \sqrt{-1},$$

$$\therefore \cos \theta = \frac{\left(1_1\right)^{\frac{\theta}{2\pi}} + \left(1_1\right)^{-\frac{\theta}{2\pi}}}{2},$$

$$\sin \theta = \frac{\left(1_1\right)^{\frac{\theta}{2\pi}} - \left(1_1\right)^{-\frac{\theta}{2\pi}}}{2 \sqrt{-1}};$$

In like manner it may be proved that

$$\sec \theta \left(1_1\right)^{\frac{\theta}{2\pi}} = 1 + \tan \theta \sqrt{-1},$$

$$\therefore \sec \theta \left(1_1\right)^{-\frac{\theta}{2\pi}} = 1 - \tan \theta \sqrt{-1},$$

$$\therefore \text{by addition } \sec \theta \left\{ \left(1_1\right)^{\frac{\theta}{2\pi}} + \left(1_1\right)^{-\frac{\theta}{2\pi}} \right\} = 2,$$

$$\text{by subtraction } \sec \theta \left\{ \left(1_1\right)^{\frac{\theta}{2\pi}} - \left(1_1\right)^{-\frac{\theta}{2\pi}} \right\} = 2 \tan \theta \sqrt{-1},$$

$$\therefore \sec \theta = \frac{2}{\left(1_1\right)^{\frac{\theta}{2\pi}} + \left(1_1\right)^{-\frac{\theta}{2\pi}}},$$

$$\therefore 2 \tan \theta \sqrt{-1} = \frac{2}{\left(1_1\right)^{\frac{\theta}{2\pi}} + \left(1_1\right)^{-\frac{\theta}{2\pi}}} \cdot \left\{ \left(1_1\right)^{\frac{\theta}{2\pi}} - \left(1_1\right)^{-\frac{\theta}{2\pi}} \right\},$$

$$\therefore \tan \theta = \frac{1}{\sqrt{-1}} \cdot \frac{\left(1_1\right)^{\frac{2\theta}{2\pi}} - 1}{\left(1_1\right)^{\frac{2\theta}{2\pi}} + 1};$$

$$\operatorname{Cosec} \theta = \sec \left(\frac{\pi}{2} - \theta \right)$$

$$= \frac{2}{\left(1_1\right)^{\frac{1}{4} - \frac{\theta}{2\pi}} + \left(1_1\right)^{-\frac{1}{4} + \frac{\theta}{2\pi}}} = \frac{2 \left(1_1\right)^{\frac{1}{4}}}{\left(1_1\right)^{\frac{1}{2} - \frac{\theta}{2\pi}} + \left(1_1\right)^{\frac{\theta}{2\pi}}}$$

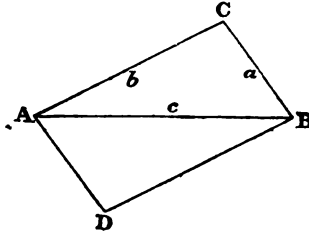
$$= \frac{2 \left(1_1\right)^{\frac{1}{4}}}{\left(1_1\right)^{\frac{\theta}{2\pi}} + \left(1_1\right)^{\frac{1}{4}} \cdot \left(1_1\right)^{-\frac{\theta}{2\pi}}} = \frac{2 \sqrt{-1}}{\left(1_1\right)^{\frac{\theta}{2\pi}} - \left(1_1\right)^{-\frac{\theta}{2\pi}}};$$

$$\operatorname{Cotan} \theta = \tan \left(\frac{\pi}{2} - \theta \right)$$

$$\begin{aligned}
\therefore \cotan \theta &= \frac{1}{\sqrt{-1}} \cdot \frac{\left(\frac{1}{1}\right)^{\frac{\pi-2\theta}{2\pi}} - 1}{\left(\frac{1}{1}\right)^{\frac{\pi-2\theta}{2\pi}} + 1} \\
&= \frac{1}{\sqrt{-1}} \cdot \frac{\left(\frac{1}{1}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{1}\right)^{-\frac{2\theta}{2\pi}} - 1}{\left(\frac{1}{1}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{1}\right)^{-\frac{2\theta}{2\pi}} + 1} \\
&= \frac{1}{\sqrt{-1}} \cdot \frac{-\left(\frac{1}{1}\right)^{-\frac{2\theta}{2\pi}} - 1}{-\left(\frac{1}{1}\right)^{-\frac{2\theta}{2\pi}} + 1} \\
&= \frac{-1}{\sqrt{-1}} \cdot \frac{1 + \left(\frac{1}{1}\right)^{\frac{2\theta}{2\pi}}}{-1 + \left(\frac{1}{1}\right)^{\frac{2\theta}{2\pi}}} \\
&= \sqrt{-1} \cdot \frac{\left(\frac{1}{1}\right)^{\frac{2\theta}{2\pi}} + 1}{\left(\frac{1}{1}\right)^{\frac{2\theta}{2\pi}} - 1}.
\end{aligned}$$

(140.) DEF. When θ is said to be equal to any angle, it is to be understood that θ is equal to the arc which subtends that angle to a radius equal unity; thus if θ be said to be equal to angle *BAF* in figure Art. 139, it is to be understood that $\theta = BF$.

(141.) *To express the relations of the sides and angles of triangles by means of powers of unity.*



Let ABC be a triangle, and let its angles be $= A, B, C$, and the opposite sides be in length respectively equal to a, b, c ;

Let AB be in the positive direction, and let the parallelogram $ACBD$ be completed, and let BAC be a positive angle;

Then $BAD = -B$, and AD is in length $= a$;

$$\therefore AC = b \left(1\right)^{\frac{A}{2\pi}}, AD = a \left(1\right)^{-\frac{B}{2\pi}};$$

$$\therefore c = b \left(1\right)^{\frac{A}{2\pi}} + a \left(1\right)^{-\frac{B}{2\pi}};$$

In like manner it may be proved that

$$b = a \left(1\right)^{\frac{C}{2\pi}} + c \left(1\right)^{-\frac{A}{2\pi}};$$

In which two equations, together with the consideration that A, B, C being angles must be each less than π , are contained all the properties of triangles.

(142.) Ex. 1. *To prove that the three angles of any triangle are together equal to π .*

$$\text{Since } b = a \left(1\right)_1^{\frac{C}{2\pi}} + c \left(1\right)_1^{-\frac{A}{2\pi}},$$

by multiplying both sides of the equation by $\left(1\right)_1^{\frac{A}{2\pi}}$,

$$b \left(1\right)_1^{\frac{A}{2\pi}} = a \left(1\right)_1^{\frac{A+C}{2\pi}} + c,$$

$$\text{But } b \left(1\right)_1^{\frac{A}{2\pi}} + a \left(1\right)_1^{-\frac{B}{2\pi}} = c,$$

$$\therefore \text{ by subtraction } - a \left(1\right)_1^{-\frac{B}{2\pi}} = a \left(1\right)_1^{\frac{A+C}{2\pi}},$$

$$\therefore -1 = \left(1\right)_1^{\frac{A+B+C}{2\pi}};$$

Now $-1 = \left(1\right)_1^{p+\frac{1}{2}}$, where p either $= 0$, or some whole number either positive or negative;

$$\therefore p + \frac{1}{2} = \frac{A+B+C}{2\pi};$$

Since $\frac{A+B+C}{2\pi}$ is positive, $p+\frac{1}{2}$ is positive,

$\therefore p$ cannot be a negative whole number;

And since, A, B, C , are each less than π ,

$$\frac{A+B+C}{2\pi} \text{ is less than } \frac{3}{2},$$

$\therefore p$ cannot be a positive whole number;

$$\therefore p = 0;$$

$$\therefore \frac{1}{2} = \frac{A+B+C}{2\pi},$$

$$\therefore A+B+C = \pi.$$

(143.) **Ex. 2.** *To prove that the sides of triangles are to one another as the sines of the opposite angles.*

First to prove that

$$b : c :: \sin B : \sin C,$$

$$\begin{aligned} c &= b \left(1\right)^{\frac{A}{2\pi}} + a \left(1\right)^{-\frac{B}{2\pi}} \\ &= b \left(1\right)^{\frac{2\pi-(B+C)}{2\pi}} + a \left(1\right)^{-\frac{B}{2\pi}} \\ &= b \cdot \left(1\right)^{\frac{1}{2}} \cdot \left(1\right)^{-\frac{B+C}{2\pi}} + a \left(1\right)^{-\frac{B}{2\pi}} \end{aligned}$$

$$= -b \left(1\right)_1^{-\frac{B+C}{2\pi}} + a \left(1\right)_1^{-\frac{B}{2\pi}},$$

$$\therefore c \left(1\right)_1^{\frac{B}{2\pi}} = -b \left(1\right)_1^{-\frac{C}{2\pi}} + a,$$

$$\therefore a = c \left(1\right)_1^{\frac{B}{2\pi}} + b \left(1\right)_1^{-\frac{C}{2\pi}},$$

$$\therefore a = c \left(1\right)_1^{-\frac{B}{2\pi}} + b \left(1\right)_1^{\frac{C}{2\pi}} \text{ (by Art. 116.)}$$

\therefore by subtraction

$$0 = c \left\{ \left(1\right)_1^{\frac{B}{2\pi}} - \left(1\right)_1^{-\frac{B}{2\pi}} \right\} - b \left\{ \left(1\right)_1^{\frac{C}{2\pi}} - \left(1\right)_1^{-\frac{C}{2\pi}} \right\},$$

$$0 = c \cdot 2 \sin B \sqrt{-1} - b \cdot 2 \sin C \sqrt{-1},$$

$$0 = c \sin B - b \sin C,$$

$$\therefore b : c :: \sin B : \sin C.$$

It may be proved in like manner that

$$a : b :: \sin A : \sin B,$$

$$\text{and } a : c :: \sin A : \sin C.$$

(144.) **Ex. 3.** *To prove that* $\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$

$$c = b \left(1\right)_1^{\frac{A}{2\pi}} + a \left(1\right)_1^{-\frac{B}{2\pi}},$$

$$\therefore a \left(1\right)^{-\frac{B}{2\pi}} = c - b \left(1\right)^{\frac{A}{2\pi}},$$

$$\therefore a \left(1\right)^{\frac{B}{2\pi}} = c - b \left(1\right)^{-\frac{A}{2\pi}},$$

\therefore by multiplication

$$a^2 = c^2 - bc \left\{ \left(1\right)^{\frac{A}{2\pi}} + \left(1\right)^{-\frac{A}{2\pi}} \right\} + b^2$$

$$= c^2 - bc \cdot 2 \cos A + b^2,$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

In like manner any other properties of triangles may be established.

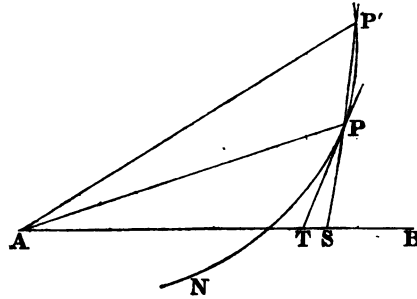
(145.) COR. *Hence all the properties of triangles are contained in the two equations*

$$c = b \left(1\right)^{\frac{A}{2\pi}} + a \left(1\right)^{-\frac{B}{2\pi}},$$

$$A + B + C = \pi.$$

For the equation $b = a \left(1\right)^{\frac{C}{2\pi}} + c \left(1\right)^{-\frac{A}{2\pi}}$ may be deduced from these two equations in the same manner that the equation $a = c \left(1\right)^{\frac{B}{2\pi}} + b \left(1\right)^{-\frac{C}{2\pi}}$ was deduced from them in Art. 143.

(146.) *To find the lengths of curves and the position of their tangents.*



Let NPP' be any curve,

A the origin, AB the positive direction,

AP , AP' two radii, PT a tangent at P ;

Let PP be joined and produced to S ;

Let AP be represented in length and direction
by ρ ,

Let AP' be represented in length and direction
by ρ'

Let NP be in length $= s$,

..... NP' $= s'$,

.....chord PP' $= k$,

Let angle $BAP = \theta$,

..... $APT = \phi$,

..... $TPS = \delta$;

Then angle $BTP = \theta + \phi$,

and $BSP = \theta + \phi + \delta$;

$$\therefore \rho' - \rho = k \cdot \left(1\right)_1^{\frac{\theta + \phi + \delta}{2\pi}},$$

$$\therefore \frac{\rho' - \rho}{s' - s} = \frac{k}{s' - s} \cdot \left(1\right)_1^{\frac{\theta + \phi + \delta}{2\pi}};$$

Now let PP' be diminished *sine limite*,

Then limit of $\frac{k}{s' - s} = 1$ (Newton, Lemma 7.)

and limit of $\left(1\right)_1^{\frac{\theta + \phi + \delta}{2\pi}} = \left(1\right)_1^{\frac{\theta + \phi}{2\pi}}$ (Newton, Lemma 6.)

$$\therefore \frac{d\rho}{ds} = \left(1\right)_1^{\frac{\theta + \phi}{2\pi}};$$

From which equation the differential expressions for determining the lengths of curves, and the position of their tangents may be deduced.

(147.) Ex. 1. *Let the curve be a circle, and A its centre.*

$\rho = r \left(1\right)_1^{\frac{\theta}{2\pi}}$, where r is a constant quantity,

$$\therefore \frac{d\rho}{ds} = r \left(1\right)_1^{\frac{\theta}{2\pi}} \cdot \frac{c}{2\pi} \cdot \sqrt{-1} \cdot \frac{d\theta}{ds} \text{ (by Art. 137.)}$$

$$\therefore r \left(1\right)_1^{\frac{\theta}{2\pi}} \cdot \frac{c}{2\pi} \cdot \sqrt{-1} \frac{d\theta}{ds} = \left(1\right)_1^{\frac{\theta+\phi}{2\pi}},$$

$$\therefore r \cdot \frac{c}{2\pi} \sqrt{-1} \frac{d\theta}{ds} = \left(1\right)_1^{\frac{\phi}{2\pi}},$$

$$\therefore -r \frac{c}{2\pi} \cdot \sqrt{-1} \cdot \frac{d\theta}{ds} = \left(1\right)_1^{-\frac{\phi}{2\pi}},$$

$$\therefore \text{by division } -1 = \left(1\right)_1^{\frac{2\phi}{2\pi}},$$

$$\therefore \frac{2\phi}{2\pi} = \frac{1}{2},$$

$$\therefore \phi = \frac{\pi}{2},$$

that is, the tangent is at right angles to the radius;

$$\therefore \left(1\right)_1^{\frac{\phi}{2\pi}} = \left(1\right)_1^{\frac{1}{4}} = +\sqrt{-1},$$

$$\therefore r \cdot \frac{c}{2\pi} \sqrt{-1} \frac{d\theta}{ds} = \sqrt{-1},$$

$$\therefore r \cdot \frac{c}{2\pi} \frac{d\theta}{ds} = 1,$$

$$\therefore s = r \cdot \frac{c}{2\pi} \theta + \text{const.}$$

When $\theta = 0$, let $s = 0$, then $\text{const.} = 0$,

$$\therefore s = r \cdot \frac{c}{2\pi} \theta.$$

(148.) COR. 1. Let $r = 1$, then $s = \text{arc}$ which subtends angle BAP to radius $= 1$, that is, $s = \theta$,

$$\therefore \frac{c}{2\pi} = 1,$$

$$\therefore 2\pi = c =$$

$$12. \left\{ \frac{1}{\sqrt{3}} - \frac{1}{3} \left(\frac{1}{\sqrt{3}} \right)^3 + \frac{1}{5} \left(\frac{1}{\sqrt{3}} \right)^5 - \&c. \right\} \text{ (Art. 129).}$$

(149.) COR. 2. Hence $s = r\theta$.

(150.) EX. 2. Let the curve be any spiral, A its pole.

Here $\rho = r \left(\frac{1}{1} \right)^{\frac{\theta}{2\pi}}$, where r is a variable quantity;

$$\therefore \rho = r \left(\frac{1}{1} \right)^{\frac{\theta}{c}} \text{ (by Art. 148.)}$$

$$\therefore \frac{d\rho}{ds} = \left(\frac{1}{1} \right)^{\frac{\theta}{c}} \cdot \frac{dr}{ds} + r \left(\frac{1}{1} \right)^{\frac{\theta}{c}} \cdot \frac{d\theta}{ds} \sqrt{-1},$$

$$\therefore \left(1\right)^{\frac{\theta}{c}} \cdot \frac{dr}{ds} + r \left(1\right)^{\frac{\theta}{c}} \cdot \frac{d\theta}{ds} \sqrt{-1} = \left(1\right)^{\frac{\theta+\phi}{c}},$$

$$\therefore \frac{dr}{ds} + r \frac{d\theta}{ds} \sqrt{-1} = \left(1\right)^{\frac{\phi}{c}},$$

$$\therefore \frac{dr}{ds} - r \frac{d\theta}{ds} \sqrt{-1} = \left(1\right)^{-\frac{\phi}{c}},$$

\therefore by multiplication,

$$\left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 = 1,$$

which is the differential equation for the length of a spiral.

Also by addition and subtraction,

$$2 \frac{dr}{ds} = \left(1\right)^{\frac{\phi}{c}} + \left(1\right)^{-\frac{\phi}{c}} = 2 \cos \phi,$$

$$2r \frac{d\theta}{ds} \sqrt{-1} = \left(1\right)^{\frac{\phi}{c}} - \left(1\right)^{-\frac{\phi}{c}} = 2 \sin \phi \sqrt{-1},$$

$$\therefore \left. \begin{array}{l} \cos \phi = \frac{dr}{ds} \\ \sin \phi = r \cdot \frac{d\theta}{ds} \\ \tan \phi = r \frac{d\theta}{dr} \end{array} \right\} \begin{array}{l} \text{equations for determining the po-} \\ \text{position of the tangent.} \end{array}$$

(151.) Ex. 3. *Let the curve be referred to rectangular co-ordinates.*

Let A be the origin,

..... x be measured in the positive direction,

..... y in the direction of $+\sqrt{-1}$;

$$\text{then } \rho = x + y \sqrt{-1},$$

$$\therefore \frac{d\rho}{ds} = \frac{dx}{ds} + \frac{dy}{ds} \sqrt{-1},$$

$$\therefore \frac{dx}{ds} + \frac{dy}{ds} \sqrt{-1} = \left(\frac{1}{1}\right)^{\frac{\theta+\phi}{c}},$$

$$\therefore \frac{dx}{ds} - \frac{dy}{ds} \sqrt{-1} = \left(\frac{1}{1}\right)^{-\frac{\theta+\phi}{c}},$$

\therefore by multiplication,

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

the differential equation for the length of a curve.

Also by addition and subtraction,

$$2 \frac{dx}{ds} = \left(\frac{1}{1}\right)^{\frac{\theta+\phi}{c}} + \left(\frac{1}{1}\right)^{-\frac{\theta+\phi}{c}} = 2 \cos (\theta + \phi),$$

$$2 \frac{dy}{ds} \sqrt{-1} = \left(\frac{1}{1}\right)^{\frac{\theta+\phi}{c}} - \left(\frac{1}{1}\right)^{-\frac{\theta+\phi}{c}} = 2 \sin (\theta + \phi) \sqrt{-1},$$

$$\therefore \cos (\theta + \phi) = \frac{dx}{ds},$$

$$\sin (\theta + \phi) = \frac{dy}{ds},$$

$$\tan (\theta + \phi) = \frac{dy}{dx};$$

And $\theta + \phi = \text{angle } BTP$ (Art. 146.)

..... = the angle at which the tangent cuts the axis;

\therefore the differential equations for determining the position of the tangent have been found.

$$(152.) \quad \text{Sin } x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$$

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$$

$$\text{For } \sin x = \frac{\left(\frac{1}{1}\right)^{\frac{\pi}{2}} - \left(\frac{1}{1}\right)^{-\frac{\pi}{2}}}{2\sqrt{-1}}$$

$$= \frac{1}{2\sqrt{-1}} \cdot \left\{ \begin{array}{l} 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. \\ -1 + x\sqrt{-1} + \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} - \frac{x^4}{1.2.3.4} + \&c. \end{array} \right\}$$

$$= x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$$

$$\cos x = \frac{\left(\frac{1}{1}\right)^{\frac{\pi}{2}} + \left(\frac{1}{1}\right)^{-\frac{\pi}{2}}}{2}$$

$$= 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$$

(153.) DEF. One *second* is assumed as the unit of time; and any other time t'' is represented by a line measured in the positive direction, and bearing the same ratio to unity which t'' does to $1''$ (see Art. 1).

(154.) DEF. The degree of swiftness or slowness with which a body is moving at any time is called its *velocity* at that time; and the velocity is thus *measured*, let t be the time, at the end of which the velocity is required, and s the length of the path described by the body in time t (the path being either a straight line or a curve, and the motion of the body in that path being either uniform, or accelerated, or retarded), let s' be the length of the path described in the time $t+h$; and let $V : 1 ::$ limiting ratio of $s' - s : h$, when h is diminished *sine limite*; then V is a line which *in length measures* the *velocity* at the end of time t ; also let the direction in which the body is moving at the

end of time t be inclined to unity at an angle $= \alpha$,
 and let $u = V \cdot \left(1\right)_1^{\frac{\alpha}{c}}$, then u is a line which in
length and *direction* measures the *velocity* at the end
 of the time t .

$$(155.) \text{ COR. 1. Hence } V = \frac{ds}{dt}, u = \frac{ds}{dt} \cdot \left(1\right)_1^{\frac{\alpha}{c}}.$$

(156.) COR. 2. Let NP in figure of Art. 146,
 be the path of the body;

$$\text{then } \alpha = \phi + \theta, \therefore u = \frac{ds}{dt} \cdot \left(1\right)_1^{\frac{\phi + \theta}{c}};$$

$$\text{But } \left(1\right)_1^{\frac{\phi + \theta}{c}} = \frac{d\rho}{ds},$$

$$\therefore u = \frac{ds}{dt} \cdot \frac{d\rho}{ds} = \frac{d\rho}{dt};$$

$$\text{Also } V^2 = \left(\frac{ds}{dt}\right)^2,$$

$$\text{and } 1 = \left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 \quad (\text{Art. 150.})$$

$$\therefore V^2 = \left(\frac{dr}{ds}\right)^2 \cdot \left(\frac{ds}{dt}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 \cdot \left(\frac{ds}{dt}\right)^2$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2;$$

$$\text{Also } 1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 \quad (\text{Art. 151.})$$

$$\begin{aligned} \therefore V^2 &= \left(\frac{dx}{ds}\right)^2 \cdot \left(\frac{ds}{dt}\right)^2 + \left(\frac{dy}{ds}\right)^2 \cdot \left(\frac{ds}{dt}\right)^2 \\ &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2. \end{aligned}$$

(157.) DEF. If the velocity of a body be instantaneously changed either in length or direction, the cause which produces this effect is called an *impulsive force* or an *impulse*; but if the velocity be gradually changed, the cause which produces this effect is called a *continued force*.

(158.) DEF. A continued force, when its effects are estimated with respect only to the change produced in the velocity, without regard to the magnitude of the body moved, is called an *accelerating force*; and is thus *measured*, let u represent in *length* and *direction* the velocity of the body at the end of time t , u' , at the end of time $t + h$, and let $f : 1 ::$ limiting ratio $u' - u : h$, when h is diminished *sine limite*; then f is a line which represents in *length* and *direction* the accelerating force at the end of time t .

$$(159.) \text{ COR. Hence } f = \frac{du}{dt} = \frac{d^2\rho}{dt^2}.$$

(160.) *When a body revolves in a circle in consequence of the action of a force tending to the centre, to determine the velocity and the periodic time.*

Let ρ represent the radius in length and direction at end of time t ,

Let r be a positive quantity in length $= \rho$,

Let ρ be inclined to unity at an angle $= \theta$,

$$\text{then } \rho = r \left(1\right)_1^{\frac{\theta}{c}},$$

$$\text{force} = \frac{d^2\rho}{dt^2},$$

And this force tends to the centre,

$\therefore \frac{d^2\rho}{dt^2}$ is a line measured in the opposite direction to that in which ρ is measured,

$\therefore \frac{d^2\rho}{dt^2}$ is inclined to unity at an angle $= \pi + \theta$,

Let $\frac{d^2\rho}{dt^2} = P \cdot \left(1\right)_1^{\frac{\pi + \theta}{c}}$, P is a positive quantity;

$$\text{now } \rho = r \left(1\right)_1^{\frac{\theta}{c}},$$

∴ since r is constant in a circle,

$$\frac{d\rho}{dt} = r \left(1\right)_1^{\frac{\theta}{c}} \cdot \frac{d\theta}{dt} \sqrt{-1},$$

$$\therefore \frac{d^2\rho}{dt^2} = -r \left(1\right)_1^{\frac{\theta}{c}} \left(\frac{d\theta}{dt}\right)^2 + r \left(1\right)_1^{\frac{\theta}{c}} \frac{d^2\theta}{dt^2} \sqrt{-1},$$

$$\therefore -r \left(1\right)_1^{\frac{\theta}{c}} \left(\frac{d\theta}{dt}\right)^2 + r \left(1\right)_1^{\frac{\theta}{c}} \frac{d^2\theta}{dt^2} \sqrt{-1} = P \cdot \left(1\right)_1^{\frac{\pi+\theta}{c}},$$

$$\therefore -r \left(\frac{d\theta}{dt}\right)^2 + r \frac{d^2\theta}{dt^2} \sqrt{-1} = P \cdot \left(1\right)_1^{\frac{\pi}{c}} = -P;$$

∴ we have the two following equations,

$$\left. \begin{aligned} -r \left(\frac{d\theta}{dt}\right)^2 &= -P \\ r \frac{d^2\theta}{dt^2} &= 0 \end{aligned} \right\};$$

$$\therefore \frac{d^2\theta}{dt^2} = 0,$$

$$\therefore \frac{d\theta}{dt} = \text{const.} = C;$$

∴ $P = r \cdot C^2$, a constant quantity;

$$\text{And } \frac{d\theta}{dt} = \sqrt{\frac{P}{r}},$$

$$\therefore t = \theta \sqrt{\frac{r}{P}} + \text{const.}$$

when $\theta = 0$, let $t = 0$, then $\text{const.} = 0$;

$$\therefore t = \theta \sqrt{\frac{r}{P}};$$

Let $\theta = 2\pi$,

$$\text{then periodic time} = 2\pi \sqrt{\frac{r}{P}};$$

Let V be in length equal to the velocity,

$$\text{then generally } V^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \text{ (by Art 156.)}$$

$$\therefore \text{in a circle } V^2 = r^2 \left(\frac{d\theta}{dt}\right)^2 = rP,$$

$$\therefore V = \sqrt{rP}.$$

(161.) *When a body is acted upon by a centripetal force, to find the sectorial area described round the centre, and the velocity.*

$$\text{As in Art. 160. } \frac{d^2\rho}{dt^2} = P \cdot \left(1\right)_1^{\frac{\pi+\theta}{c}},$$

$$\text{and } \rho = r \left(1\right)_1^{\frac{\theta}{c}};$$

$$\therefore \frac{d\rho}{dt} = \frac{dr}{dt} \cdot \left(1\right)_1^{\frac{\theta}{c}} + r \left(1\right)_1^{\frac{\theta}{c}} \frac{d\theta}{dt} \sqrt{-1},$$

$$\begin{aligned}\therefore \frac{d^2 \rho}{dt^2} &= \frac{d^2 r}{dt^2} \cdot \left(1\right)_1^{\frac{\theta}{c}} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \cdot \left(1\right)_1^{\frac{\theta}{c}} \sqrt{-1} \\ &\quad - r \left(\frac{d\theta}{dt}\right)^2 \cdot \left(1\right)_1^{\frac{\theta}{c}} + r \frac{d^2 \theta}{dt^2} \cdot \left(1\right)_1^{\frac{\theta}{c}} \sqrt{-1},\end{aligned}$$

$$\begin{aligned}\therefore P \cdot \left(1\right)_1^{\frac{\pi+\theta}{c}} &= \frac{d^2 r}{dt^2} \cdot \left(1\right)_1^{\frac{\theta}{c}} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \cdot \left(1\right)_1^{\frac{\theta}{c}} \sqrt{-1} \\ &\quad - r \left(\frac{d\theta}{dt}\right)^2 \cdot \left(1\right)_1^{\frac{\theta}{c}} + r \frac{d^2 \theta}{dt^2} \cdot \left(1\right)_1^{\frac{\theta}{c}} \sqrt{-1},\end{aligned}$$

$$\begin{aligned}\therefore P \cdot \left(1\right)_1^{\frac{\pi}{c}} &= \frac{d^2 r}{dt^2} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \cdot \sqrt{-1} \\ &\quad - r \left(\frac{d\theta}{dt}\right)^2 + r \frac{d^2 \theta}{dt^2} \cdot \sqrt{-1};\end{aligned}$$

$$\text{but } P \left(1\right)_1^{\frac{\pi}{c}} = -P,$$

$$\begin{aligned}\therefore -P &= \frac{d^2 r}{dt^2} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \cdot \sqrt{-1} \\ &\quad - r \left(\frac{d\theta}{dt}\right)^2 + r \frac{d^2 \theta}{dt^2} \cdot \sqrt{-1};\end{aligned}$$

\therefore we have the two following equations,

$$\begin{aligned}(1) \quad &\left. \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2 = -P \right\}, \\ (2) \quad &2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} = 0.\end{aligned}$$

multiplying all the terms of (2) by r , we have

$$2r \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r^2 \frac{d^2\theta}{dt^2} = 0,$$

\therefore integrating, $r^2 \frac{d\theta}{dt} = h$, a constant quantity,

$$\therefore \int r^2 d\theta = ht + \text{const.}$$

when $\int r^2 d\theta = 0$, let $t = t'$,

then $\int r^2 d\theta = h(t - t')$,

$$\therefore \int \frac{r^2 d\theta}{2} = \frac{h}{2}(t - t'),$$

or sectorial area = $\frac{h}{2}(t - t')$.

Now multiplying every term of (1) by $2 \cdot \frac{dr}{dt}$ and

every term of (2) by $2r \frac{d\theta}{dt}$, we have

$$\left. \begin{aligned} 2 \frac{dr}{dt} \cdot \frac{d^2r}{dt^2} - 2r \frac{dr}{dt} \cdot \left(\frac{d\theta}{dt}\right)^2 &\dots\dots\dots = -2P \frac{dr}{dt} \\ \dots\dots\dots 4r \frac{dr}{dt} \cdot \left(\frac{d\theta}{dt}\right)^2 + 2r^2 \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} &= 0 \end{aligned} \right\};$$

\therefore by addition,

$$2 \frac{dr}{dt} \cdot \frac{d^2r}{dt^2} + 2r \frac{dr}{dt} \cdot \left(\frac{d\theta}{dt}\right)^2 + 2r^2 \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} = -2P \frac{dr}{dt};$$

\therefore integrating, $\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \dots\dots\dots = f - 2Pdr,$

\therefore (by Art. 156.) $V^2 \dots\dots\dots = f - 2Pdr.$

(162.) COR. 1. From equation (1),

$$\frac{d^2r}{dt^2} = r \left(\frac{d\theta}{dt}\right)^2 - P$$

= centrifugal force – centripetal force.

(163.) COR. 2. Since $r^2 \frac{d\theta}{dt} = h,$

$$\left(\frac{dt}{d\theta}\right)^2 = \frac{r^4}{h^2};$$

$$\text{and } \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 = V^2,$$

$$\therefore \left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{r^4 V^2}{h^2},$$

$$\therefore V^2 = h^2 \left\{ \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} \right\}.$$

(164.) COR. 3. Let $u = \frac{1}{r},$

$$\text{then } \frac{du}{d\theta} = -\frac{1}{r^2} \cdot \frac{dr}{d\theta},$$

$$\therefore V^2 = h^2 \left\{ \left(\frac{du}{d\theta}\right)^2 + u^2 \right\}.$$

(165.) COR. 4. Since $V^2 = f - 2P dr$,

$$V^2 = f + \frac{2P du}{u^2},$$

$$\therefore h^2 \left\{ \left(\frac{du}{d\theta} \right)^2 + u^2 \right\} = \int \frac{2P du}{u^2},$$

$$\therefore \text{differentiating, } h^2 \left\{ 2 \frac{du}{d\theta} \cdot \frac{d^2 u}{d\theta^2} + 2u \frac{du}{d\theta} \right\} = \frac{2P}{u^2} \frac{du}{d\theta},$$

$$\therefore h^2 \left\{ \frac{d^2 u}{d\theta^2} + u \right\} = \frac{P}{u^2},$$

$$\therefore \frac{d^2 u}{d\theta^2} + u - \frac{P}{h^2 u^2} = 0.$$

(166.) Let $P \propto \frac{1}{r^2}$, to find the equation to the curve described.

$$\text{Let } P = \frac{m}{r^2} = mu^2;$$

$$\therefore \frac{d^2 u}{d\theta^2} + u - \frac{m}{h^2} = 0, \text{ (by Art. 165.)}$$

$$\text{Let } u - \frac{m}{h^2} = x,$$

$$\therefore \frac{du}{d\theta} = \frac{dx}{d\theta},$$

$$\frac{d^2 u}{d\theta^2} = \frac{d^2 x}{d\theta^2},$$

$$\therefore \frac{d^2 x}{d\theta^2} + x = 0;$$

Let $(e)^{k\theta} \cdot \left(\frac{1}{1}\right)^{\frac{n\theta}{c}}$ represent a particular value of x ,
where e is the base of the hyperbolic logarithms,

$$\text{then } \frac{dx}{d\theta} = (e)^{k\theta} \cdot \left(\frac{1}{1}\right)^{\frac{n\theta}{c}} (k + n\sqrt{-1}),$$

$$\frac{d^2 x}{d\theta^2} = (e)^{k\theta} \cdot \left(\frac{1}{1}\right)^{\frac{n\theta}{c}} \cdot (k + n\sqrt{-1})^2;$$

$$\therefore (e)^{k\theta} \cdot \left(\frac{1}{1}\right)^{\frac{n\theta}{c}} \cdot (k + n\sqrt{-1})^2 + (e)^{k\theta} \cdot \left(\frac{1}{1}\right)^{\frac{n\theta}{c}} = 0,$$

$$\therefore (k + n\sqrt{-1})^2 + 1 = 0,$$

$$\therefore k + n\sqrt{-1} = \pm \sqrt{-1},$$

$$\therefore k = 0, \quad n = \pm 1;$$

\therefore the general value is,

$$x = C \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + C' \left(\frac{1}{1}\right)^{-\frac{\theta}{c}};$$

Let $C = a \left(\frac{1}{1}\right)^{\frac{\alpha}{c}}$, $C' = b \left(\frac{1}{1}\right)^{\frac{\beta}{c}}$, where a , b are
positive quantities and α , β positive and less than
 2π ;

$$\text{then } x = a \left(\frac{1}{1}\right)^{\frac{\alpha+\theta}{c}} + b \left(\frac{1}{1}\right)^{\frac{\beta-\theta}{c}},$$

$$x = a \left(1_1\right)^{\frac{\alpha+\theta}{c}} + b \left(1_1\right)^{\frac{\beta-\theta}{c}}$$

$$\therefore x = a \left(1_1\right)^{-\frac{\alpha+\theta}{c}} + b \left(1_1\right)^{-\frac{\beta-\theta}{c}};$$

\therefore by subtraction,

$$\begin{aligned} 0 &= a \cdot \left(1_1\right)^{\frac{\alpha+\theta}{c}} - a \cdot \left(1_1\right)^{-\frac{\alpha+\theta}{c}} + b \cdot \left(1_1\right)^{\frac{\beta-\theta}{c}} - b \left(1_1\right)^{-\frac{\beta-\theta}{c}} \\ &= \left\{ a \left(1_1\right)^{\frac{\alpha}{c}} - b \left(1_1\right)^{-\frac{\beta}{c}} \right\} \cdot \left(1_1\right)^{\frac{\theta}{c}} - \left\{ a \left(1_1\right)^{-\frac{\alpha}{c}} - b \cdot \left(1_1\right)^{\frac{\beta}{c}} \right\} \cdot \left(1_1\right)^{-\frac{\theta}{c}}, \end{aligned}$$

$$\therefore 0 = \left\{ a \left(1_1\right)^{\frac{\alpha}{c}} - b \left(1_1\right)^{-\frac{\beta}{c}} \right\} \cdot \left(1_1\right)^{\frac{2\theta}{c}} - \left\{ a \cdot \left(1_1\right)^{-\frac{\alpha}{c}} - b \cdot \left(1_1\right)^{\frac{\beta}{c}} \right\};$$

\therefore since θ is variable, we have the two following equations,

$$\left. \begin{aligned} a \left(1_1\right)^{\frac{\alpha}{c}} - b \left(1_1\right)^{-\frac{\beta}{c}} &= 0 \\ a \left(1_1\right)^{-\frac{\alpha}{c}} - b \left(1_1\right)^{\frac{\beta}{c}} &= 0 \end{aligned} \right\};$$

from either of which we deduce

$$b = a$$

$$\left(1_1\right)^{\frac{\beta}{c}} = \left(1_1\right)^{-\frac{\alpha}{c}},$$

\therefore substituting these values for b and $\left(1\right)^{\frac{\theta}{c}}$,

$$x = a \left(1\right)^{\frac{\alpha+\theta}{c}} + a \left(1\right)^{-\frac{\alpha+\theta}{c}},$$

$$= a \cdot 2 \cos (\alpha + \theta);$$

$$\therefore u - \frac{m}{h^2} = 2a \cos (\alpha + \theta),$$

$$\therefore \frac{1}{r} - \frac{m}{h^2} = 2a \cos (\alpha + \theta),$$

equation to the curve described.

(167.) *To solve a cubic equation $x^3 - qx + r = 0$, when $\frac{q^3}{27}$ is greater than $\frac{r^2}{4}$.*

By Cardan's rule,

$$\begin{aligned} x &= \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}} \\ &= \left\{-\frac{r}{2} + \left(\frac{q^3}{27} - \frac{r^2}{4}\right)^{\frac{1}{2}} \sqrt{-1}\right\}^{\frac{1}{3}} \\ &\quad + \left\{-\frac{r}{2} - \left(\frac{q^3}{27} - \frac{r^2}{4}\right)^{\frac{1}{2}} \sqrt{-1}\right\}^{\frac{1}{3}}; \end{aligned}$$

$$\text{Let } \left\{ -\frac{r}{2} + \left(\frac{q^3}{27} - \frac{r^2}{4} \right)^{\frac{1}{2}} \sqrt{-1} \right\}^{\frac{1}{3}} = f \left(\frac{1}{1} \right)^{\frac{\theta}{c}},$$

where f is a positive quantity, and $\frac{\theta}{c}$ positive and less than 1;

$$\text{Then } \left\{ -\frac{r}{2} - \left(\frac{q^3}{27} - \frac{r^2}{4} \right)^{\frac{1}{2}} \sqrt{-1} \right\}^{\frac{1}{3}} = f \left(\frac{1}{1} \right)^{-\frac{\theta}{c}},$$

$$\therefore x = f \left(\frac{1}{1} \right)^{\frac{\theta}{c}} + f \left(\frac{1}{1} \right)^{-\frac{\theta}{c}} = f \cdot 2 \cos \theta;$$

By multiplication,

$$\left(\frac{r^2}{4} + \frac{q^3}{27} - \frac{r^2}{4} \right)^{\frac{1}{2}} = f^2;$$

$$\therefore f = \sqrt{\frac{q}{3}};$$

$$\therefore x = 2 \sqrt{\frac{q}{3}} \cdot \cos \theta;$$

$$\text{Also } -\frac{r}{2} + \left(\frac{q^3}{27} - \frac{r^2}{4} \right)^{\frac{1}{2}} \cdot \sqrt{-1} = f^3 \left(\frac{1}{1} \right)^{\frac{3\theta}{c}},$$

$$\text{And } -\frac{r}{2} - \left(\frac{q^3}{27} - \frac{r^2}{4} \right)^{\frac{1}{2}} \cdot \sqrt{-1} = f^3 \left(\frac{1}{1} \right)^{-\frac{3\theta}{c}};$$

$$\begin{aligned} \therefore -r \dots\dots\dots &= f^3 \left\{ \left(\frac{1}{1} \right)^{\frac{3\theta}{c}} + \left(\frac{1}{1} \right)^{-\frac{3\theta}{c}} \right\} \\ &= f^3 \cdot 2 \cos (3\theta); \end{aligned}$$

$$\therefore \cos (3\theta) = -\frac{r}{2f^3} = -\frac{r}{2\left(\frac{q}{3}\right)^{\frac{3}{2}}} = -\frac{r\sqrt{27}}{2\sqrt{q^3}}.$$

Let $-\frac{r\sqrt{27}}{2\sqrt{q^3}} = \cos \phi$, where ϕ is positive and less than 560° ,

Then ϕ is known from trigonometrical tables ;

$$\text{And since } -\frac{r\sqrt{27}}{2\sqrt{q^3}} = \cos \phi,$$

$$\begin{aligned} -\frac{r\sqrt{27}}{2\sqrt{q^3}} &= \cos (360^\circ + \phi), \\ &= \cos (2.360^\circ + \phi), \end{aligned}$$

$$\therefore \cos (3\theta) = \cos \phi = \cos (360^\circ + \phi) = \cos (2.360^\circ + \phi);$$

\therefore the values of θ are

$$\frac{\phi}{3}, 120^\circ + \frac{\phi}{3}, 240^\circ + \frac{\phi}{3};$$

\therefore the values of x are

$$\begin{aligned} 2\sqrt{\frac{q}{3}} \cdot \cos \left(\frac{\phi}{3}\right), 2\sqrt{\frac{q}{3}} \cdot \cos \left(120^\circ + \frac{\phi}{3}\right), \\ 2\sqrt{\frac{q}{3}} \cdot \cos \left(240^\circ + \frac{\phi}{3}\right), \end{aligned}$$

which may be found from trigonometrical tables.

(168.) To find the integral of $\frac{dx}{x}$.

$$\text{Let } x = (e)^{my} \cdot \left(\frac{1}{1}\right)^{\frac{ny}{e}},$$

$$\begin{aligned} \text{Then } \frac{dx}{dy} &= (e)^{my} \cdot \left(\frac{1}{1}\right)^{\frac{ny}{e}} \cdot (m + n\sqrt{-1}), \\ &= x \cdot (m + n\sqrt{-1}), \end{aligned}$$

$$\therefore \frac{1}{x} \cdot \frac{dx}{dy} = m + n\sqrt{-1},$$

$$\therefore \int \frac{dx}{x} = my + ny\sqrt{-1};$$

$$\text{Now } x \text{ is in length} = (e)^{my},$$

$\therefore my$ is the hyperbolic logarithm of the length of x ;

And ny is the angle, at which x is inclined to unity,

\therefore if this angle be called *angle of x* ,

$$\int \frac{dx}{x} = \text{hyp. log. of length of } x + \sqrt{-1} \cdot \text{angle of } x.$$

(169.) To find the integral of $\frac{2adx}{a^2 + x^2}$.

$$\begin{aligned}
\frac{2a}{a^2+x^2} &= \frac{1}{a+x\sqrt{-1}} + \frac{1}{a-x\sqrt{-1}}, \\
\therefore \int \frac{2adx}{a^2+x^2} &= \int \frac{dx}{a+x\sqrt{-1}} + \int \frac{dx}{a-x\sqrt{-1}} \\
&= \frac{1}{\sqrt{-1}} \cdot \int \frac{dx\sqrt{-1}}{a+x\sqrt{-1}} - \frac{1}{\sqrt{-1}} \cdot \int \frac{-dx\sqrt{-1}}{a-x\sqrt{-1}} \\
&= \frac{1}{\sqrt{-1}} \left\{ \text{hyp. log. of length of } \overline{a+x\sqrt{-1}} \right. \\
&\quad \left. + \sqrt{-1} \cdot \text{angle of } \overline{a+x\sqrt{-1}} \right\} \\
&\quad - \frac{1}{\sqrt{-1}} \left\{ \text{hyp. log. of length of } \overline{a-x\sqrt{-1}} \right. \\
&\quad \left. + \sqrt{-1} \cdot \text{angle of } \overline{a-x\sqrt{-1}} \right\} \\
&= \frac{1}{\sqrt{-1}} \cdot \left\{ \text{hyp. log. of length of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} \right. \\
&\quad \left. + \sqrt{-1} \cdot \text{angle of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} \right\} \\
&= \frac{1}{\sqrt{-1}} \cdot \left\{ \text{hyp. log. of length of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} \right. \\
&\quad \left. + \text{angle of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} \right\}.
\end{aligned}$$

(170.) COR. Hence, if a and x be positive quantities, and v be an angle whose tangent is $\frac{x}{a}$;

$$\int \frac{2a dx}{a^2 + x^2} = 2v.$$

For in this case $\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}$ is in length = 1,

$$\therefore \text{hyp. log. of length of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} = 0;$$

And angle of $\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} = \text{angle of } \frac{(a+x\sqrt{-1})^2}{a^2 + x^2},$

$$= \text{angle of } \left(1 + \frac{x}{a}\sqrt{-1}\right)^2,$$

$$= 2 \cdot \text{angle of } \left(1 + \frac{x}{a}\sqrt{-1}\right),$$

$$= 2v;$$

$$\therefore \int \frac{2adx}{a^2 + x^2} = 2v.$$

(171.) To find the integral of $\frac{dx}{\pm \sqrt{x^2 + a^2}}.$

$$\text{Let } \pm \sqrt{x^2 + a^2} = y,$$

$$\text{then } x^2 + a^2 = y^2,$$

$$\text{and } \frac{dy}{dx} = \frac{x}{y},$$

$$\therefore \frac{dy+dx}{dx} = \frac{x+y}{y},$$

$$\therefore \int \frac{dx}{y} = \int \frac{dx+dy}{x+y},$$

$$\text{or } \int \frac{dx}{\pm \sqrt{x^2+a^2}} = \int \frac{dx+dy}{x+y}$$

$$= \text{hyp. log. of length of } (x+y)$$

$$+ \sqrt{-1} \cdot \text{angle of } (x+y)$$

$$= \text{hyp. log. of length of } (x \pm \sqrt{x^2+a^2})$$

$$+ \sqrt{-1} \cdot \text{angle of } (x \pm \sqrt{x^2+a^2}).$$

(172.) COR. In like manner the integral of

$$\frac{dx}{\pm \sqrt{x^2+2ax}} \text{ will be found to be}$$

$$\text{hyp. log. of length of } (x+a \pm \sqrt{x^2+2ax})$$

$$+ \sqrt{-1} \cdot \text{angle of } (x+a \pm \sqrt{x^2+2ax});$$

And by similar investigations the integrals of other logarithmic and circular functions may be obtained.

(173.) *When a body is moving about a centre of force as in Art. 161; a radius vector, and the velocity of the body at the extremity of that radius vector being known both in length and direction; to find the value of the constant quantity h.*

Let $V \cdot \left(1_1\right)^{\frac{\phi}{c}}$ be the velocity,

$$\text{then } V \cdot \left(1_1\right)^{\frac{\phi}{c}} = \frac{d\rho}{dt}$$

$$= \left(1_1\right)^{\frac{\theta}{c}} \cdot \frac{dr}{dt} + r \left(1_1\right)^{\frac{\theta}{c}} \cdot \frac{d\theta}{dt} \cdot \sqrt{-1},$$

$$\therefore V \cdot \left(1_1\right)^{\frac{\phi-\theta}{c}} = \frac{dr}{dt} + r \frac{d\theta}{dt} \sqrt{-1},$$

$$\text{or } V \cdot \cos (\phi-\theta) + V \cdot \sin (\phi-\theta) \cdot \sqrt{-1}$$

$$= \frac{dr}{dt} + r \cdot \frac{d\theta}{dt} \cdot \sqrt{-1},$$

$$\therefore V \sin (\phi-\theta) = r \cdot \frac{d\theta}{dt};$$

$$\text{But } h = r^2 \cdot \frac{d\theta}{dt},$$

$$\therefore h = r V \cdot \sin (\phi-\theta).$$

S

(174.) *Let a body be projected from the extremity of a given radius vector, with a given velocity, in a direction perpendicular to the radius vector, and let it be acted upon by a force tending to the origin of the radii vectors, and varying inversely as the square of the distance; to find the equation to the curve described by the body, and the time of describing any part of that curve.*

The equation to the curve described in this case has already been found in Article 166; but, in order further to illustrate the principles established in the preceding chapters, another method of finding this equation is here given;

Let the given radius vector be in the positive direction, and let it = l ,

Let the velocity of a body revolving in a circle at the distance l with the same centripetal force be in length = V ,

And let the velocity of projection be in length = nV ;

Then velocity of projection = $nV\sqrt{-1}$;

Also $V = \sqrt{P.l}$, (by Art. 160.)

$$= \sqrt{\frac{ml}{l^2}} = \sqrt{\frac{m}{l}},$$

$$\therefore m = lV^2,$$

$$\text{Also } h = l.nV \text{ (by Art. 173);}$$

$$\text{Now } \frac{d^2\rho}{dt^2} = -\frac{m}{r^2} \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} = -\frac{l.V^2}{r^2} \left(\frac{1}{1}\right)^{\frac{\theta}{c}},$$

$$\text{and } \frac{1}{r^2} = \frac{1}{h} \cdot \frac{d\theta}{dt} = \frac{1}{lnV} \cdot \frac{d\theta}{dt},$$

$$\therefore \frac{d^2\rho}{dt^2} = -\frac{V}{n} \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} \cdot \frac{d\theta}{dt},$$

$$\therefore \frac{d\rho}{dt} = -\frac{V}{n} \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} \cdot \frac{1}{\sqrt{-1}} + \text{const.}$$

$$= \frac{V}{n} \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} \cdot \sqrt{-1} + \text{const.}$$

$$\text{when } \theta = 0, \frac{d\rho}{dt} = nV \cdot \sqrt{-1},$$

$$\therefore nV\sqrt{-1} = \frac{V}{n} \sqrt{-1} + \text{const.},$$

$$\therefore \text{const.} = \frac{n^2-1}{n} \cdot V \cdot \sqrt{-1},$$

$$\text{Let } n^2-1 = e,$$

$$\text{then const.} = \frac{eV}{n} \sqrt{-1};$$

$$\therefore \frac{d\rho}{dt} = \frac{V}{n} \sqrt{-1} \cdot \left\{ e + \left(\frac{1}{1} \right)^{\frac{\theta}{c}} \right\};$$

$$\text{Now } \frac{1}{r^2} = \frac{1}{\ln V} \cdot \frac{d\theta}{dt},$$

$$\therefore \frac{1}{\rho^2} = \frac{1}{\ln V} \cdot \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} \frac{d\theta}{dt},$$

$$\begin{aligned} \therefore \frac{1}{\rho^2} \cdot \frac{d\rho}{dt} &= \frac{\sqrt{-1}}{n^2 \cdot l} \cdot \left\{ e + \left(\frac{1}{1} \right)^{\frac{\theta}{c}} \right\} \cdot \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} \cdot \frac{d\theta}{dt} \\ &= \frac{\sqrt{-1}}{1+e \cdot l} \cdot \left\{ e \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} \cdot \frac{d\theta}{dt} + \left(\frac{1}{1} \right)^{-\frac{\theta}{c}} \frac{d\theta}{dt} \right\}, \end{aligned}$$

$$\begin{aligned} \therefore -\frac{1}{\rho} &= \frac{\sqrt{-1}}{1+e \cdot l} \cdot \left\{ \frac{e \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}}}{-2\sqrt{-1}} + \frac{\left(\frac{1}{1} \right)^{-\frac{\theta}{c}}}{-\sqrt{-1}} \right\} + \text{const.} \\ &= -\frac{1}{2 \cdot (1+e) \cdot l} \left\{ e \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} + 2 \left(\frac{1}{1} \right)^{-\frac{\theta}{c}} \right\} + \text{const.}, \end{aligned}$$

When $\theta = 0$, $\rho = l$,

$$\therefore -\frac{1}{l} = -\frac{e+2}{2(1+e) \cdot l} + \text{const.},$$

$$\therefore \text{const.} = \frac{e+2-2-2e}{2(1+e) \cdot l} = -\frac{e}{2(1+e) \cdot l},$$

$$\therefore \frac{1}{\rho} = \frac{1}{2(1+e)l} \cdot \left\{ e + e \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} + 2 \left(\frac{1}{1} \right)^{-\frac{\theta}{c}} \right\},$$

$$\begin{aligned}
 \therefore \frac{1}{r} &= \frac{1}{2(1+e)l} \cdot \left\{ e \left(\frac{1}{1} \right)^{\frac{\theta}{c}} + e \left(\frac{1}{1} \right)^{-\frac{\theta}{c}} + 2 \right\} \\
 &= \frac{1}{2(1+e)l} \cdot (2e \cos \theta + 2) \\
 &= \frac{1}{(1+e)l} \cdot (e \cos \theta + 1),
 \end{aligned}$$

$$\text{Let } a = \frac{l}{1-e},$$

$$\text{Then } \frac{1}{r} = \frac{1+e \cos \theta}{a \cdot (1-e^2)}, \text{ equation to curve described.}$$

Next to find the time of describing any part of this curve,

$$\begin{aligned}
 \frac{1}{\rho} &= \frac{2}{2(1+e)l} \cdot \left\{ e + e \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} + 2 \left(\frac{1}{1} \right)^{-\frac{\theta}{c}} \right\} \\
 &= \frac{1}{2(1-e^2) \cdot a} \left\{ e + e \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} + 2 \left(\frac{1}{1} \right)^{-\frac{\theta}{c}} \right\}, \\
 \therefore \frac{2(1-e^2)}{e} \cdot \frac{a}{\rho} &= 1 + \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} + \frac{2}{e} \cdot \left(\frac{1}{1} \right)^{-\frac{\theta}{c}}, \\
 \therefore \left(\frac{1}{1} \right)^{-\frac{2\theta}{c}} + \frac{2}{e} \cdot \left(\frac{1}{1} \right)^{-\frac{\theta}{c}} + \frac{1}{e^2} &= \frac{2 \cdot (1-e^2)}{e} \frac{a}{\rho} + \frac{1}{e^2} - 1 \\
 &= \frac{2(1-e^2)}{e} \cdot \frac{a}{\rho} + \frac{1-e^2}{e^2}
 \end{aligned}$$

$$\therefore \left(1\right)_1^{-\frac{2\theta}{c}} + \frac{2}{e} \left(1\right)_1^{-\frac{\theta}{c}} + \frac{1}{e^2} = \frac{(\rho + 2ae) \cdot (1 - e^2)}{\rho e^2},$$

$$\therefore \left(1\right)_1^{-\frac{\theta}{c}} + \frac{1}{e} = \pm \frac{(\rho + 2ae)^{\frac{1}{2}} \cdot (1 - e^2)^{\frac{1}{2}}}{e \rho^{\frac{1}{2}}},$$

$$\therefore \left(1\right)_1^{-\frac{\theta}{c}} = \pm \frac{(\rho + 2ae)^{\frac{1}{2}} \cdot (1 - e^2)^{\frac{1}{2}}}{e \rho^{\frac{1}{2}}} - \frac{1}{e};$$

$$\text{Now } \frac{d\rho}{dt} = \frac{V}{n} \sqrt{-1} \cdot \left\{ e + \left(1\right)_1^{\frac{\theta}{c}} \right\},$$

$$\therefore \frac{dt}{d\rho} = \frac{n}{V \sqrt{-1}} \cdot \frac{1}{e + \left(1\right)_1^{\frac{\theta}{c}}}$$

$$= \frac{n}{V \sqrt{-1}} \cdot \frac{\left(1\right)_1^{-\frac{\theta}{c}}}{e \left(1\right)_1^{-\frac{\theta}{c}} + 1}$$

$$= \frac{n}{e V \sqrt{-1}} \cdot \frac{\left(1\right)_1^{-\frac{\theta}{c}}}{\left(1\right)_1^{-\frac{\theta}{c}} + \frac{1}{e}}$$

$$= \frac{n}{e V \sqrt{-1}} \cdot \left\{ 1 \mp \frac{\rho^{\frac{1}{2}}}{(\rho + 2ae)^{\frac{1}{2}} \cdot (1 - e^2)^{\frac{1}{2}}} \right\},$$

$$\therefore t = \frac{\sqrt{1+e}}{e V \cdot \sqrt{1-e^2} \cdot \sqrt{-1}} \cdot \left\{ (1-e^2)^{\frac{1}{2}} \rho \mp \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho + 2ae)^{\frac{1}{2}}} \right\} + \text{const.},$$

$$\therefore t = \frac{1}{eV \cdot \sqrt{1-e} \cdot \sqrt{-1}} \cdot \left\{ (1-e^2)^{\frac{1}{2}} \rho \mp \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho+2ae)^{\frac{1}{2}}} \right\} + \text{const.};$$

$$\text{Now } \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho+2ae)^{\frac{1}{2}}}$$

$$= 2\rho^{\frac{1}{2}}(\rho+2ae)^{\frac{1}{2}} - \int (\rho+2ae)^{\frac{1}{2}} \cdot \frac{d\rho}{\rho^{\frac{1}{2}}}$$

$$= 2\rho^{\frac{1}{2}}(\rho+2ae)^{\frac{1}{2}} - \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho+2ae)^{\frac{1}{2}}} - \int \frac{2aed\rho}{\rho^{\frac{1}{2}}(\rho+2ae)^{\frac{1}{2}}},$$

$$\therefore \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho+2ae)^{\frac{1}{2}}} = \rho^{\frac{1}{2}} \cdot (\rho+2ae)^{\frac{1}{2}} - \int \frac{aed\rho}{\rho^{\frac{1}{2}}(\rho+2ae)^{\frac{1}{2}}},$$

$$\therefore \mp \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho+2ae)^{\frac{1}{2}}} = \mp (\rho^2+2ae\rho)^{\frac{1}{2}} \pm ae \int \frac{d\rho}{(\rho^2+2ae\rho)^{\frac{1}{2}}}$$

$$= \mp (\rho^2+2ae\rho)^{\frac{1}{2}} + ae \int \frac{d\rho}{\pm (\rho^2+2ae\rho)^{\frac{1}{2}}}$$

$$= \mp (\rho^2+2ae\rho)^{\frac{1}{2}}$$

$$+ ae \cdot \text{hyp. log. of length of } (\rho+ae \pm \sqrt{\rho^2+2ae\rho})$$

$$+ ae \sqrt{-1} \cdot \text{angle of } (\rho+ae \pm \sqrt{\rho^2+2ae\rho});$$

$$\therefore t = \frac{1}{eV \cdot \sqrt{1-e} \cdot \sqrt{-1}} \cdot \left\{ (1-e^2)^{\frac{1}{2}} \rho \mp (\rho^2+2ae\rho)^{\frac{1}{2}} \right.$$

$$+ ae \cdot \text{hyp. log. of length of } (\rho+ae \pm \sqrt{\rho^2+2ae\rho})$$

$$\left. + ae \sqrt{-1} \cdot \text{angle of } (\rho+ae \pm \sqrt{\rho^2+2ae\rho}) \right\} + \text{const.};$$

$$\text{Now } \pm \frac{(\rho + 2ae)^{\frac{1}{2}} \cdot (1 - e^2)^{\frac{1}{2}}}{e\rho^{\frac{1}{2}}} = \left(\frac{1}{1}\right)^{-\frac{\theta}{c}} + \frac{1}{e},$$

$$\therefore \pm (\rho^2 + 2ae\rho)^{\frac{1}{2}} = \rho \cdot \frac{1 + e \left(\frac{1}{1}\right)^{-\frac{\theta}{c}}}{(1 - e^2)^{\frac{1}{2}}};$$

$$\text{When } t=0, \theta=0, \rho = \overline{1 - e} \cdot a,$$

$$\text{and } \pm (\rho^2 + 2ae\rho)^{\frac{1}{2}} = \frac{(1 - e) \cdot a \cdot (1 + e)}{(1 - e^2)^{\frac{1}{2}}} = a \cdot (1 - e^2)^{\frac{1}{2}};$$

$$\therefore 0 = \frac{1}{eV \cdot \sqrt{1 - e} \cdot \sqrt{-1}} \cdot \left\{ (1 - e^2)^{\frac{1}{2}} \cdot (1 - e)a - a \cdot (1 - e^2)^{\frac{1}{2}} \right.$$

$$+ ae \cdot \text{hyp. log. of length of } (a + a\sqrt{1 - e^2})$$

$$\left. + ae \cdot \sqrt{-1} \cdot \text{angle of } (a + a\sqrt{1 - e^2}) \right\} + \text{const.}$$

$$= \frac{1}{eV \sqrt{1 - e} \cdot \sqrt{-1}} \cdot \left\{ - (1 - e^2)^{\frac{1}{2}} \cdot ae \right.$$

$$+ ae \cdot \text{hyp. log. of length of } (a + a\sqrt{1 - e^2})$$

$$\left. + ae \cdot \sqrt{-1} \cdot \text{angle of } (a + a\sqrt{1 - e^2}) \right\} + \text{const.,}$$

$$\therefore t = \frac{1}{eV \sqrt{1 - e} \cdot \sqrt{-1}} \cdot \left\{ \sqrt{1 - e^2} \cdot (\rho + ae) \mp (\rho^2 + 2ae\rho)^{\frac{1}{2}} \right.$$

$$+ ae \cdot \text{hyp. log. of length of } \frac{\rho + ae \pm \sqrt{\rho^2 + 2ae\rho}}{a + a\sqrt{1 - e^2}}$$

$$\left. + ae \cdot \sqrt{-1} \cdot \text{angle of } \frac{\rho + ae \pm \sqrt{\rho^2 + 2ae\rho}}{a + a\sqrt{1 - e^2}} \right\},$$

an equation which gives t in terms of ρ , where that value of $\pm \sqrt{\rho^2 + 2ae\rho}$ must be used, which satisfies the conditions of the equation

$$\frac{\pm (\rho + 2ae)^{\frac{1}{2}}}{e\rho^{\frac{1}{2}}} (1 - e^2)^{\frac{1}{2}} = \left(\frac{1}{1}\right)^{-\frac{\theta}{c}} + \frac{1}{e}.$$

(175.) *To find the time t in terms of the angle θ .*

$$\mp \sqrt{\rho^2 + 2ae\rho} = -\rho \cdot \frac{1 + e \left(\frac{1}{1}\right)^{-\frac{\theta}{c}}}{(1 - e^2)^{\frac{1}{2}}}$$

$$= -\frac{r \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + er}{(1 - e^2)^{\frac{1}{2}}},$$

$$\therefore \sqrt{1 - e^2} \cdot (\rho + ae) \mp (\rho^2 + 2ae\rho)^{\frac{1}{2}}$$

$$= \frac{(1 - e^2) \cdot \left\{ r \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + ae \right\} - r \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} - er}{(1 - e^2)^{\frac{1}{2}}}$$

$$= \frac{e}{(1 - e^2)^{\frac{1}{2}}} \cdot \left\{ (1 - e^2) \cdot a - r \cdot \left[1 + e \left(\frac{1}{1}\right)^{\frac{\theta}{c}} \right] \right\}$$

$$= \frac{e}{(1 - e^2)^{\frac{1}{2}}} \cdot \left\{ (1 - e^2) a - \frac{(1 - e^2) \cdot a \cdot \left\{ 1 + e \left(\frac{1}{1}\right)^{\frac{\theta}{c}} \right\}}{1 + e \cos \theta} \right\}$$

T

$$\begin{aligned}
& \therefore \sqrt{1-e^2} \cdot (\rho + ae) \mp \sqrt{\rho^2 + 2ae\rho} \\
&= ae \cdot (1-e^2)^{\frac{1}{2}} \cdot \frac{1+e \cos \theta - 1 - e \left(\frac{1}{1}\right)^{\frac{\theta}{c}}}{1+e \cos \theta} \\
&= ae^2 (1-e^2)^{\frac{1}{2}} \cdot \frac{\cos \theta - \cos \theta - \sin \theta \sqrt{-1}}{1+e \cos \theta} \\
&= - \frac{ae^2 \cdot (1-e^2)^{\frac{1}{2}} \cdot \sin \theta \sqrt{-1}}{1+e \cos \theta};
\end{aligned}$$

$$\text{Also } \rho + ae \pm \sqrt{\rho^2 + 2ae\rho}$$

$$\begin{aligned}
&= r \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + ae + \frac{r \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + er}{(1-e^2)^{\frac{1}{2}}} \\
&= ae + r \frac{(\sqrt{1-e^2} + 1) \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + e}{\sqrt{1-e^2}} \\
&= ae + a \cdot \sqrt{1-e^2} \cdot \frac{(\sqrt{1-e^2} + 1) \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + e}{1+e \cos \theta} \\
&= a \cdot \frac{e + e^2 \cos \theta + (1-e^2 + \sqrt{1-e^2}) \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + e\sqrt{1-e^2}}{1+e \cos \theta} \\
&= a \cdot \frac{e + e\sqrt{1-e^2} + e^2 \cos \theta + (1-e^2 + \sqrt{1-e^2}) \cdot (\cos \theta + \sin \theta \sqrt{-1})}{1+e \cos \theta} \\
&= a \cdot \frac{e + e\sqrt{1-e^2} + (1 + \sqrt{1-e^2}) \cdot \cos \theta + (1-e^2 + \sqrt{1-e^2}) \sin \theta \sqrt{-1}}{1+e \cos \theta}
\end{aligned}$$

$$= a.(1 + \sqrt{1-e^2}). \frac{e + \cos \theta + \sqrt{1-e^2} \cdot \sin \theta \sqrt{-1}}{1 + e \cos \theta},$$

$$\therefore \frac{\rho + ae \pm \sqrt{\rho^2 + 2ae\rho}}{a + a\sqrt{1-e^2}} = \frac{e + \cos \theta + \sqrt{1-e^2} \cdot \sin \theta \sqrt{-1}}{1 + e \cos \theta};$$

$$\therefore t = \frac{1}{e.V.\sqrt{1-e}.\sqrt{-1}} \cdot \left\{ ae. \text{hyp. log. of length of} \right.$$

$$\frac{e + \cos \theta + \sqrt{1-e^2} \cdot \sin \theta \cdot \sqrt{-1}}{1 + e \cos \theta}$$

$$+ ae.\sqrt{-1} \text{ angle of } \frac{e + \cos \theta + \sqrt{1-e^2} \cdot \sin \theta \cdot \sqrt{-1}}{1 + e \cos \theta}$$

$$\left. - \frac{ae^2 \sqrt{1-e^2} \cdot \sin \theta \cdot \sqrt{-1}}{1 + e \cos \theta} \right\}$$

$$= \frac{a}{V.\sqrt{1-e}} \cdot \left\{ \frac{1}{\sqrt{-1}} \cdot \text{hyp. log. of length of} \right.$$

$$\frac{e + \cos \theta + \sqrt{1-e^2} \cdot \sin \theta \sqrt{-1}}{1 + e \cos \theta}$$

$$+ \text{angle of } \frac{e + \cos \theta + \sqrt{1-e^2} \cdot \sin \theta \cdot \sqrt{-1}}{1 + e \cos \theta}$$

$$\left. - \frac{e\sqrt{1-e^2} \cdot \sin \theta}{1 + e \cos \theta} \right\};$$

$$\text{now } V = \sqrt{\frac{m}{l}},$$

$$\therefore V \cdot \sqrt{1-e} = \sqrt{\frac{m \cdot 1-e}{l}} = \sqrt{\frac{m}{a}};$$

$$\therefore \frac{a}{V \sqrt{1-e}} = \frac{a^{\frac{3}{2}}}{\sqrt{m}},$$

$$\therefore t = \frac{a^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ \frac{1}{\sqrt{-1}} \cdot \text{hyp. log. of length of} \right.$$

$$\frac{e + \cos \theta + \sqrt{1-e^2} \cdot \sin \theta \cdot \sqrt{-1}}{1 + e \cos \theta}$$

$$+ \text{angle of } \frac{e + \cos \theta + \sqrt{1-e^2} \cdot \sin \theta \cdot \sqrt{-1}}{1 + e \cos \theta}$$

$$\left. - \frac{e(1-e^2)^{\frac{1}{2}} \cdot \sin \theta}{1 + e \cos \theta} \right\},$$

an equation which gives the value of t in terms of θ .

(176.) *To determine what this value of t becomes according as different values are given to n .*

First let n be less than 1,

Then e is negative,

Let $e = -e'$,

then $t = \frac{a^{\frac{1}{2}}}{\sqrt{m}} \left\{ \frac{1}{\sqrt{-1}} \cdot \text{hyp. log of length of} \right.$

$$\frac{-e' + \cos \theta + \sqrt{1-e'^2} \cdot \sin \theta \sqrt{-1}}{1 - e' \cos \theta}$$

+ angle of $\frac{-e' + \cos \theta + \sqrt{1-e'^2} \cdot \sin \theta \sqrt{-1}}{1 - e' \cos \theta}$

$$+ \frac{e' \sqrt{1-e'^2} \cdot \sin \theta}{1 - e' \cos \theta} \left. \right\},$$

Now length of $\frac{-e' + \cos \theta + \sqrt{1-e'^2} \cdot \sin \theta \sqrt{-1}}{1 - e' \cos \theta}$

$$= \frac{\{(-e' + \cos \theta)^2 + (1 - e'^2) \cdot \sin^2 \theta\}^{\frac{1}{2}}}{1 - e' \cos \theta} = 1,$$

\therefore hyp. log. of length of

$$\frac{-e' + \cos \theta + \sqrt{1-e'^2} \cdot \sin \theta \sqrt{-1}}{1 - e' \cos \theta} = 0,$$

and angle of $\frac{-e' + \cos \theta + \sqrt{1-e'^2} \cdot \sin \theta \sqrt{-1}}{1 - e' \cos \theta},$

$$= \text{angle, whose cosine} = \frac{-e' + \cos \theta}{1 - e' \cos \theta},$$

$$\text{and whose sine} = \frac{\sqrt{1-e'^2} \cdot \sin \theta}{1 - e' \cos \theta},$$

Let this angle = u ,

$$\text{Then } t = \frac{a^{\frac{1}{2}}}{\sqrt{m}} \cdot \left\{ u + e' \sin u \right\}.$$

Next let $n = 1$,

Then $e = 0$,

$$\begin{aligned} \therefore t &= \frac{a^{\frac{1}{2}}}{\sqrt{m}} \left\{ \frac{1}{\sqrt{-1}} \cdot \text{hyp. log. of length of} \right. \\ &\quad (\cos \theta + \sin \theta \sqrt{-1}) \\ &\quad \left. + \text{angle of } (\cos \theta + \sin \theta \sqrt{-1}) \right\} \\ &= \frac{a^{\frac{1}{2}}}{\sqrt{m}} \cdot \left\{ \frac{1}{\sqrt{-1}} \cdot \text{hyp. log. of length of } \left(1\right)^{\frac{\theta}{c}} \right. \\ &\quad \left. + \text{angle of } \left(1\right)^{\frac{\theta}{c}} \right\} \\ &= \frac{a^{\frac{1}{2}}}{\sqrt{m}} \cdot \theta. \end{aligned}$$

Next let n be greater than 1 and less than $\sqrt{2}$,

Then e is positive and less than 1 ;

$$\text{Let } u \text{ be an angle whose cosine} = \frac{e + \cos \theta}{1 + e \cos \theta},$$

Then it may be proved nearly as in the first case that

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ u - e \sin u \right\}.$$

Next let $n = \sqrt{2}$,

Then $e = 1$;

Here we must find t by the method of determining the value of fractions, whose numerators and denominators are evanescent;

By the preceding case, when e is less than 1,

$$\begin{aligned} t &= \frac{a^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ u - e \sin u \right\} \\ &= \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \frac{u - e \sin u}{(1 - e)^{\frac{3}{2}}} \\ &= \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \frac{\sin u + \frac{1}{2} \cdot \frac{1}{3} \sin^3 u + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} \sin^5 u + \&c. - e \sin u}{(1 - e)^{\frac{3}{2}}} \\ &= \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \frac{(1 - e) \sin u + \frac{1}{6} \cdot \sin^3 u + \frac{3}{40} \cdot \sin^5 u + \&c.}{(1 - e)^{\frac{3}{2}}}; \\ \text{Now } \sin u &= \frac{\sqrt{1 - e^2} \cdot \sin \theta}{1 + e \cdot \cos \theta} = \frac{\sqrt{1 - e} \cdot \sqrt{1 + e} \cdot \sin \theta}{1 + e \cos \theta}, \end{aligned}$$

$$\therefore t = \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ \frac{\sqrt{1+e} \cdot \sin \theta}{1+e \cos \theta} + \frac{1}{6} \frac{(1+e)^{\frac{3}{2}} \sin^3 \theta}{(1+e \cos \theta)^3} \right. \\ \left. + \frac{3}{40} \cdot (1-e) \cdot \frac{(1+e)^{\frac{5}{2}} \sin^5 \theta}{(1+e \cos \theta)^5} + \&c. \right\};$$

Let $e = 1$,

$$\text{Then } t = \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ \sqrt{2} \cdot \frac{\sin \theta}{1 + \cos \theta} + \frac{2^{\frac{3}{2}}}{6} \cdot \frac{\sin^3 \theta}{(1 + \cos \theta)^3} \right\} \\ = \frac{l^{\frac{3}{2}} \sqrt{2}}{\sqrt{m}} \cdot \left\{ \frac{\sin \theta}{1 + \cos \theta} + \frac{1}{3} \cdot \frac{\sin^3 \theta}{(1 + \cos \theta)^3} \right\} \\ = \frac{l^{\frac{3}{2}} \sqrt{2}}{\sqrt{m}} \cdot \left\{ \tan \frac{\theta}{2} + \frac{1}{3} \cdot \tan^3 \frac{\theta}{2} \right\}.$$

Next let n be greater than $\sqrt{2}$,

Then e is greater than 1,

$$\text{In this case } a^{\frac{3}{2}} = \left(\frac{l}{1-e} \right)^{\frac{3}{2}} = \left(\frac{l}{-(e-1)} \right)^{\frac{3}{2}} \\ = \frac{l^{\frac{3}{2}}}{-\sqrt{-1} \cdot (e-1)^{\frac{3}{2}}} = \frac{l^{\frac{3}{2}} \sqrt{-1}}{(e-1)^{\frac{3}{2}}};$$

$$\text{Let } \frac{l}{e-1} = a',$$

$$\text{Then } a^{\frac{3}{2}} = a'^{\frac{3}{2}} \cdot \sqrt{-1};$$

$$\begin{aligned}
\therefore t &= \frac{a'^{\frac{1}{2}} \sqrt{-1}}{\sqrt{m}} \cdot \left\{ \frac{1}{\sqrt{-1}} \cdot \text{hyp. log. of length of} \right. \\
&\quad \frac{e + \cos \theta + \sqrt{e^2 - 1} \cdot \sqrt{-1} \cdot \sin \theta \cdot \sqrt{-1}}{1 + e \cos \theta} \\
&+ \text{angle of } \frac{e + \cos \theta + \sqrt{e^2 - 1} \cdot \sqrt{-1} \cdot \sin \theta \cdot \sqrt{-1}}{1 + e \cos \theta} \\
&\quad \left. \frac{-e \sqrt{e^2 - 1} \cdot \sqrt{-1} \cdot \sin \theta}{1 + e \cos \theta} \right\} \\
&= \frac{a'^{\frac{1}{2}}}{\sqrt{m}} \cdot \left\{ \text{hyp. log. of length of } \frac{e + \cos \theta - \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta} \right. \\
&\quad + \sqrt{-1} \cdot \text{angle of } \frac{e + \cos \theta - \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta} \\
&\quad \left. + \frac{e \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta} \right\};
\end{aligned}$$

Now $\frac{e + \cos \theta - \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta}$ is a positive quantity,

$$\therefore \text{hyp. log. of length of } \frac{e + \cos \theta - \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta}$$

$$= \text{hyp. log. } \frac{e + \cos \theta - \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta},$$

$$\text{and angle of } \frac{e + \cos \theta - \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta} = 0;$$

U

$$\therefore t = \frac{a'^{\frac{2}{3}}}{\sqrt{m}} \cdot \left\{ \text{hyp. log. } \frac{e + \cos \theta - \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta} \right. \\ \left. + \frac{e \sqrt{e^2 - 1} \cdot \sin \theta}{1 + e \cos \theta} \right\}.$$





